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THE ELEMENTS

OF

PLANE GEOMETRY.

Association for the
Improvement of Geometrical Teaching.

THE

ELEMENTS OF PLANE GEOMETRY.

PART II.

(CORRESPONDING TO EUCLID, BOOKS III., IV., V., VI.)

Prepared by the Committee appointed by the Association.

> "Proficiency in Pure Geometry . . . perennially a symptom, not only of steady application, but of a clear methodic intellect, and offering in all epochs good promise for all manner of arts and pursuits."
>
> T. CARLYLE.

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PART II.

(CORRESPON[illegible] [illegible] [illegible] BOOKS III., IV., V., VI.)

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PREFACE.

THE Second Part of the Elements of Plane Geometry, now submitted to the Public, has been prepared on the same lines as the First Part issued two years ago. Books III., IV., V., of the "Syllabus of Plane Geometry" have been revised, Demonstrations of the Propositions supplied, and suitable Exercises inserted.

The two parts contain the portion of Plane Geometry treated of in Euclid, Books I. to VI., with some additional matter, and afford a sufficient course for School Teaching. It will probably be found in most cases advantageous to postpone the study of Part II. of Book IV. until after a first reading of Book V.

The Association hopes shortly to publish Syllabuses of Elementary Solid Geometry, and Geometrical Conic Sections, and probably also of Higher Geometry, indicating courses of study in these Subjects suitable for school purposes.

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THE ELEMENTS OF PLANE GEOMETRY.

BOOK III.

THE CIRCLE.

SECTION I.

ELEMENTARY PROPERTIES.

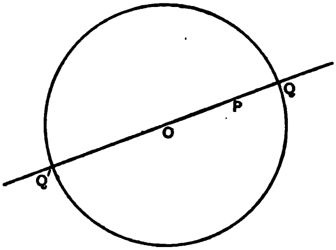
DEF. 1. *A* **circle** *is a plane figure contained by one line, which is called the* **circumference,** *and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the* **centre** *of the circle.*

DEF. 2. *A* **radius** *of a circle is a straight line drawn from the centre to the circumference.*

DEF. 3. *A* **diameter** *of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.*

THEOR. 1. **The distance of a point from the centre of a circle is less than, equal to, or greater than the radius, according as the point is within, on, or without the circumference.**

Let O be the centre of the circle,
P any other point:



then shall OP be less than, equal to, or greater than the radius, according as P is within, on, or without the circumference.

The straight line passing through O and P will meet the circumference in two points Q, Q′, and in no other points, since there are only two points on the line whose distances from O are equal to the radius.

If P is between Q and Q′, P is within the circumference, and OP is less than OQ, that is, less than the radius.

If P coincide with Q or Q′, P is on the circumference, and OP is equal to OQ or OQ′, that is, equal to the radius.

If P is on OQ or OQ′ produced, P is without the circumference, and OP is greater than OQ or OQ′, that is, greater than the radius.

Q.E.D.

Cor. **A point is within, on, or without the circumference of a circle, according as its distance from the centre is less than, equal to, or greater than the radius.**

Two points in a straight line equidistant from a point O in it, and on opposite sides of O, are said to be *symmetrical* with respect to the point O.

When to every point in a figure there corresponds another point of it symmetrical with the first with respect to a certain point O, the figure is said to be *symmetrical* or to have *point-symmetry* with respect to O, which is then called its *centre of symmetry*, or simply its *centre*.

Hence a circle is symmetrical with respect to its centre; or, the centre of a circle is a centre of symmetry.

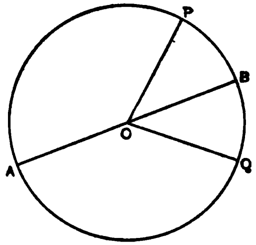
Ex. 1. Shew that a circle has only one centre.

Ex. 2. Shew that every parallelogram has point-symmetry with respect to the intersection of its diagonals.

Ex. 3. If a quadrilateral has point-symmetry with respect to the intersection of its diagonals, it must be a parallelogram.

THEOR. 2. **Any diameter of a circle divides it into two identically equal parts, called semicircles.**

Let O be the centre, and AOB any diameter of the circle APBQ, dividing it into two parts APB, AQB:



then shall APB be identically equal to AQB.

Take P any point in the arc APB, and join OP.
Take OQ a radius on the other side of AOB, and making with OB the angle BOQ equal to BOP. *I. Prob.* 5.

Then, if APB be turned about AB till it lies on the same side of AB with AQB, and in the same plane with it,
OP will fall along OQ, since the angle BOP is equal to the angle BOQ,
and P will fall on Q, since OP is equal to OQ.
Hence every point in the arc APB will coincide with some point in the arc AQB.

In like manner every point in the arc AQB may be made to coincide with some point in the arc APB.

Therefore the figure APB may be made to coincide with the figure AQB, and therefore is identically equal to it.

Q.E.D.

COR. **Any two diameters at right angles to one another divide the circle into four identically equal parts, called quadrants.**

Any line which divides a figure into two parts which are identically equal, so that if one part be turned about the line it will come to coincide with the other, is said to divide the figure *symmetrically*, and the line is said to be an *axis of symmetry.*

Hence Theor. 2 may be thus expressed:

A circle is symmetrical with respect to any one of its diameters; or, Every diameter of a circle is an axis of symmetry.

Ex. 4. Shew that the bisector of the vertical angle of an isosceles triangle is an axis of symmetry of the triangle.

Ex. 5. Shew that a rhombus is symmetrical with respect to each of its diagonals.

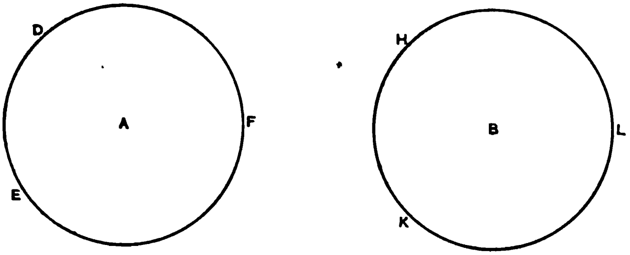
Ex. 6. If a parallelogram is symmetrical with respect to one of its diagonals, it must be a rhombus.

Ex. 7. If a quadrilateral is symmetrical with respect to each of its diagonals, it must be a rhombus.

**Ex.* 8. Prove that if a curve is symmetrical with respect to every axis through a given point, it is a circle whose centre is that point.

THEOR. 3. **Circles of equal radii are identically equal.**

Let DEF, HKL be two circles of equal radii:



then shall the circle DEF be identically equal to the circle HKL.

Let A, B be the centres of the circles DEF, HKL, respectively.

Place the circle DEF upon the circle HKL, so that the centre A may coincide with the centre B.

Then any point on the circumference of DEF will fall on the circumference of HKL, since its distance from the centre of HKL is a radius of the circle DEF, and therefore equal to the radius of HKL. *III.* 1, *Cor.*

In like manner, any point on the circumference of HKL will fall on the circumference of DEF.

Hence the two circumferences coincide,
and therefore the circles coincide,
and therefore are identically equal.

Q.E.D.

COR. 1. **If two circles coincide, and one of them is turned through any angle about their common centre, they will continue to coincide.**

COR. 2. **Concentric circles of unequal radii cannot meet.**

COR. 3. **Two circles whose circumferences meet one another cannot be concentric.**

EXERCISES.

*9. If from any point within a circle two straight lines be drawn making equal angles with the diameter through the point and on opposite sides of the diameter, the two straight lines are equal.

10. Construct an equilateral triangle having two of its vertices on a given circle, and the third at a given point within the circle.

11. Construct an isosceles triangle of given vertical angle, having its vertex at a given point within the circle and the extremities of its base on the circumference.

12. The lines joining any fixed point to any number of points on the circumference of a given circle are bisected. Shew that the middle points all lie on the circumference of another fixed circle.

13. If lines are drawn to meet a fixed circle from an external point, prove that the vertices of equilateral triangles described upon them as bases will be on two fixed circles.

Enunciate and prove a similar theorem as regards squares.

14. A point is taken outside a fixed circle at a distance from the centre not greater than three times the radius. Shew that a straight line may be drawn through the point to cut the circle in such a way that the part of the line between the point and the circle is equal to the part intercepted by the circle.

15. If two straight lines be drawn through any point on a diagonal of a square parallel to the sides of the square, the points where these lines meet the sides will lie on the circumference of a circle whose centre is at the intersection of the diagonals.

SECTION II.

ANGLES AT THE CENTRE AND SECTORS.

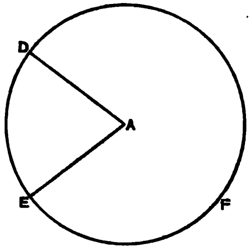
DEF. 4 *An* **arc** *is a part of a circumference. Two arcs, which together make the whole circumference, are said to be* **conjugate.** *The greater of the two is called the* **major conjugate** *arc and the smaller the* **minor conjugate** *arc.*

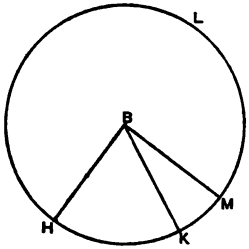
DEF. 5. *The conjugate angles formed at the centre of a circle by two radii are said to* **stand** *upon the conjugate arcs opposite them intercepted by the radii, the major angle upon the major arc, and the minor angle upon the minor arc.*

DEF. 6. *A* **sector** *is a figure contained by an arc and the radii drawn to its extremities. The* **angle of the sector** *is the angle at the centre which stands upon the arc of the sector.*

THEOR. 4. **In the same circle, or in equal circles, equal angles at the centre stand on equal arcs, and of two unequal angles at the centre the greater angle stands on the greater arc.**

In the equal circles DEF, HKL, of which A and B are t respective centres, let the angle DAE be equal to the angle HBK :





then shall the arc DE on which the angle DAE stands be equal to the arc HK on which the angle HBK stands.

Let the circle DEF be applied to the circle HKL, so that the centre A falls on the centre B,
then their circumferences will coincide, since the circles are equal. *III.* 3.

Let DEF be turned about the centre until AD falls on BH,
then the circles continue to coincide, *III.* 3, *Cor.* 1.
and the point D falls on H.

Then, because the angle DAE is equal to the angle HBK, *Hyp.*
therefore AE will fall on BK, and the point E on the point K;
therefore the arc DE coincides with, and is therefore equal to, the arc HK.

Again, if the angle DAE is greater than the angle HBK:
then shall the arc DE be greater than the arc HK.
For in this case, if the radius AD be brought to coincide with BH,
AE will coincide with some radius BM beyond BK,
and the arc DE with the arc HM, which is greater than HK;
therefore the arc DE is greater than the arc HK. Q.E.D.

Cor. Sectors of the same, or of equal circles, which have equal angles are equal, and of two such sectors which have unequal angles the greater is that which has the greater angle.

Ex. 16. If in equal circles one angle at the centre is double another, the arc on which the first angle stands is double that on which the second stands.

Ex. 17. If in equal circles the angle of one sector is double that of another, then will the first sector be double the second.

THEOR. 5. **In the same circle, or in equal circles, equal arcs subtend equal angles at the centre, and of two unequal arcs the greater subtends the greater angle at the centre.**

In the equal circles DEF, HKL of which A and B are the respective centres, let DE and HK be arcs, subtending at the centre the angles DAE, HBK:
then according as the arc DE is greater than, equal to, or less than, the arc HK, so shall the angle DAE be greater than, equal to, or less than, the angle HBK.

For in the preceding Theorem it has been shown that
if the angle DAE is greater than the angle HBK,
then the arc DE is greater than the arc HK;
if the angle DAE is equal to the angle HBK,
then the arc DE is equal to the arc HK;
if the angle DAE is less than the angle HBK,
then the arc DE is less than the arc HK.

Now of these hypotheses one must be true, and of the conclusions no two can be true at the same time;
hence, by the Rule of Conversion, the converse of each of the above Theorems is true.

Q.E.D.

COR. **Equal sectors of the same, or of equal circles, have equal angles, and of two unequal sectors the greater has the greater angle.**

EXERCISES.

18. Bisect a given arc of a circle whose centre is given.

19. Bisect a given sector.

20. Trisect a given semicircle. *I., Ex.* 81.

21. Trisect a given quadrant.

22. Trisect the circumference of a given circle whose centre is given.

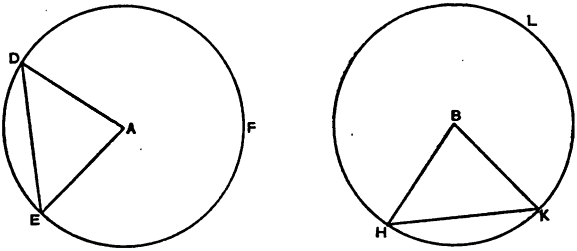
SECTION III.

CHORDS.

DEF. 7. *A* **chord** *of a circle is the straight line joining any two points on the circumference.*

THEOR. 6. **In the same circle, or in equal circles, equal arcs are subtended by equal chords; and of two unequal minor arcs the greater is subtended by the greater chord.**

In the equal circles DEF, HKL let the arc DE be equal to the arc HK:



then shall the chord DE be equal to the chord HK.

Let A and B be the centres of the circles;
join AD, AE, BH, BK.

Then, because the arc DE is equal to the arc HK, therefore, whether DE and HK be minor or major arcs, the

angle DAE of the triangle ADE is equal to the angle HBK of the triangle BHK; *Ax. e.*
III. 5, *and I.* 1.
hence in the triangles ADE, BHK,
the side AD is equal to the side BH,
the side AE is equal to the side BK,
and the angle DAE is equal to the angle HBK;
therefore the side DE is equal to the side HK. *I.* 5.

Next, let the minor arc DE be greater than the minor arc HK:
then shall the chord DE be greater than the chord HK.

Because the minor arc DE is greater than the minor arc HK:
therefore the angle DAE of the triangle ADE is greater than the angle HBK of the triangle BHK; *III.* 5.
hence in the triangles ADE, BHK,
the side AD is equal to the side BH,
the side AE is equal to the side BK,
and the angle DAE is greater than the angle HBK;
therefore the base DE is greater than the base HK. *I.* 16.
Q.E.D.

COR. **In the same circle, or in equal circles, of two unequal major arcs the greater is subtended by the less chord.**

Ex. 23. If in equal circles one arc is double another, the chord which subtends the first arc is less than the double of the chord which subtends the second arc.

THEOR. 7. **In the same circle, or in equal circles, equal chords subtend equal major and equal minor arcs; and of two unequal chords the greater subtends the greater minor arc and the less major arc.**

Let DE, HK be chords of the equal circles DEF, HKL: then according as the chord DE is greater than, equal to, or less than, the chord HK, so shall the minor arc DE be greater than equal to, or less than, the minor arc HK, and the major arc DE, be less than, equal to, or greater than, the major arc HK.

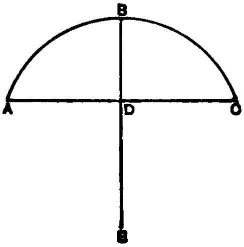
For in the preceding Theorem it has been shewn that
if the minor arc DE is greater than the minor arc HK,
then the chord DE is greater than the chord HK;
if the minor arc DE is equal to the minor arc HK,
then the chord DE is equal to the chord HK;
if the minor arc DE is less than the minor arc HK,
then the chord DE is less than the chord HK.

Now of these hypotheses one must be true, and of the conclusions no two can be true at the same time;
hence, by the Rule of Conversion, according as the chord DE is greater than, equal to, or less than, the chord HK, so is the minor arc DE greater than, equal to, or less than, the minor arc HK;
and, therefore, the whole circumferences DEF, HKL being equal the major arc DE less than, equal to, or greater than, the major arc HK.

Q.E.D.

PROB. 1. **To bisect a given arc.**

Let ABC be the given arc: it is required to bisect it.



Join AC.

Draw the straight line DE bisecting the line AC at right angles. *I. Prob.* 4.

Let DE meet the arc ABC at B;
then shall the arc ABC be bisected at B.

Because DE is the locus of points equidistant from A and C; *Loci, iii.*

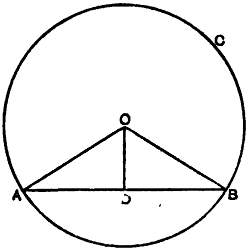
therefore the chord AB is equal to the chord BC;
therefore the arcs AB and BC, being both minor arcs, are equal; *III.* 7.

that is, the arc ABC is bisected at B.

Q.E.F.

THEOR. 8. **The straight line drawn from the centre to the middle point of a chord is perpendicular to the chord.**

Let O be the centre of the circle ABC, D the middle point of a chord AB:

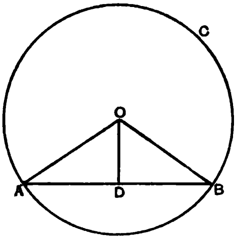


then shall the straight line OD be perpendicular to the chord AB. Join OA, OB.

Then in the triangles OAD, OBD,
the side AD is equal to the side BD, *Hyp.*
the side OD is common to both,
and the side OA is equal to the side OB; *III. Def.* 1.
therefore the angle ODA is equal to the angle ODB; *I.* 18.
therefore OD is perpendicular to AB. Q.E.D.

THEOR. 9. **The straight line drawn from the centre perpendicular to a chord bisects the chord.**

Let O be the centre of the circle ABC, OD the perpendicular from O to the chord AB:



then shall AB be bisected at D.

Join OA, OB.

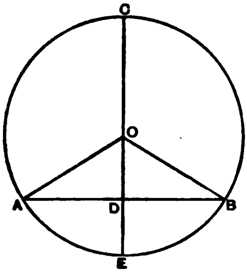
Then in the triangles OAD, OBD,
the side OA is equal to the side OB, *III. Def.* 1.
the side OD is common to both,
and the angle ODA, ODB are right angles;
therefore AD is equal to BD; *I.* 20, *Cor.* 1.
that is, AB is bisected at D. Q.E.D.

**Ex.* 24. The locus of the middle points of parallel chords in a circle is a diameter of the circle.

**Ex.* 25. A straight line passing through a common point of two intersecting circles has one extremity on each circumference. Shew that the projection on it of the line joining the centres is half the first straight line.

THEOR. 10. **The straight line drawn perpendicular to a chord through its middle point passes through the centre.**

Let AB be a chord of the circle ABC, CE the straight line drawn perpendicular to AB through D, its middle point:



then shall CE pass through the centre.

Let O be the centre of the circle.
Join OA, OB, OD.

Then in the triangles OAD, OBD,
the side AD is equal to the side BD, *Hyp.*
the side OD is common to both,
and the side OA is equal to the side OB; *III. Def.* 1.
therefore the angle ODA is equal to the angle ODB; *I.* 18.
therefore OD is perpendicular to AB.
But CD is perpendicular to AB, *Hyp.*
therefore OD coincides with CD,
that is, CD passes through the centre.

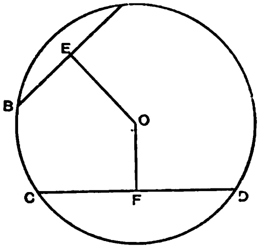
Q.E.D.

Any one of Theors. 8, 9, 10 being proved geometrically, the other two follow by the Rule of Identity.

**Ex.* 26. The locus of the centres of all circles that pass through two given points is the straight line that bisects at right angles the line joining those points.

PROB. 2. **To find the centre of a given circle, or of a given arc.**

Let ABC be the given circumference or arc of a circie: it is required to find the centre of the circle.



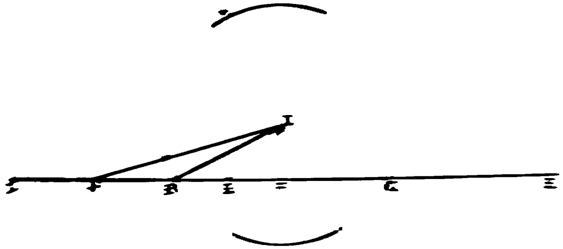
Take any two chords AB, CD, which are not parallel. Bisect AB and CD at right angles by the straight lines EO, FO meeting at O, *I. Prob.* 4.
then shall O be the centre required.

For, because EO bisects the chord AB at right angles, therefore the centre is in EO; *III.* 10.
and, because FO bisects the chord CD at right angles, therefore the centre is in FO; *III.* 10.
therefore O, the only point common to EO and FO, is the centre of the circle.

Q.E.F.

THEOR. 11. **All points in a chord of a circle between the extremities of the chord lie within the circumference; and al points in the chord produced either way lie without the circumference.**

Let ABC be a circle, DBCE a straight line through B and C, any two points on the circumference:



then shall all points in BC between B and C lie within the circumference; and all points in BD or CE, except B and C, shall lie without the circumference.

Let O be the centre of the circle, F the middle point of BC, G any point between F and B, H any point in BD: join OB, OF, OG, OH.

Because F is the middle point of the chord BC,
therefore OF is perpendicular to BC, *III.* 8.
and therefore OF is less than OB; *I.* 15.
therefore F is within the circumference. *III.* 1, *Cor.*

Again, because the angle GOF is less than the angle BOF,
therefore OG is less than OB, *I.* 15.
therefore G is within the circumference; *III.* 1, *Cor.*
and in like manner any point between F and C is within the circumference.

Also, because the angle HOF is greater than the angle BOF,
therefore OH is greater than OB; *I.* 15.
therefore H is without the circumference; *III.* 1, *Cor.*

and in like manner any point in CE, except C, is without the circumference.

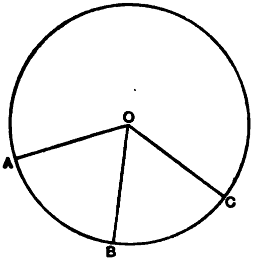
Q.E.D.

COR. **A straight line cannot meet the circumference of a circle in more than two points.**

DEF. 8. *A* **secant** *is a straight line of unlimited length which meets the circumference of a circle in two points.*

THEOR. 12. **There is one circle, and only one, whose circumference passes through three given points not in the same straight line.**

Let A, B, C be three given points not in the same straight line:



there is one circle, and only one, whose circumference passes through A, B, C.

Take O the point equidistant from A, B, and C; *Int. of Loci, i*
join OA, OB, OC.

Then, because OA, OB, OC are all equal,
therefore the circumference of a circle drawn with centre O and radius OA will pass through A, B, C.

Again, because there is only one point equidistant from A, B, C, *Int. of Loci, i.*
and because circles with a common centre and common radius coincide wholly,
therefore there is only one circle whose circumference passes through A, B, and C.

Q.E.D.

COR. 1. **Two circles whose circumferences have three points in common coincide wholly.**

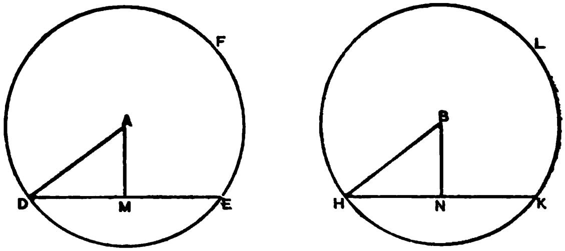
COR. 2. **The circumferences of two circles which do not coincide cannot meet in more than two points.**

COR. 3. **If from any point within a circle more than two straight lines drawn to the circumference are equal, that point is the centre.**

Ex. 27. Prove that two circles cannot have a common arc.

THEOR. 13. **In the same circle, or in equal circles, equal chords are equally distant from the centre; and of two unequal chords the greater is nearer to the centre than the less.**

In the equal circles DEF, HKL, whose centres are A and B, let the chord DE be equal to the chord HK, and let AM, BN be perpendicular to DE and HK:



then shall DE and HK be equally distant from the centres, that is, AM shall be equal to BN.

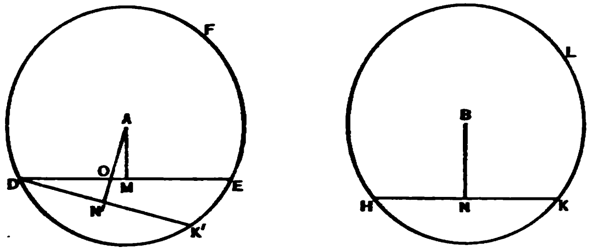
Join AD, BH.

Because AM is drawn from the centre A perpendicular to the chord DE,

therefore DM is half DE ; *III.* 9.
and in like manner HN is half HK.
But DE is equal to KH ; *Hyp.*
therefore DM is equal to HN. *Ax. h.*

Hence, in the triangles ADM, BHN,
the side DM is equal to the side HN,
the side AD is equal to the side BH,
and the angles AMD, BNH are right angles ;
therefore the side AM is equal to the side BN ; *I.* 20, *Cor.* 1.
that is, DE and HK are equally distant from the centres.

Next let the chord DE be greater than the chord HK :
then shall AM be less than BN.



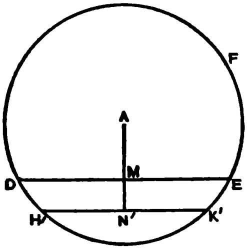
Apply the circle HKL to the circle DEF so that B coincides with A, and H with D, and the minor arc HK falls along the minor arc DE.

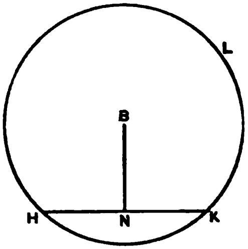
Then, because the chord HK is less than the chord DE, *Hyp.*
therefore the minor arc HK is less than the minor arc DE. *III.* 7.
Therefore K will fall upon the minor arc DE.
Let it fall at K':
then every point in the chord DK' except D is on the side of DE remote from A.
Let N' be the new position of N,
and let AN' cut DE in O.
Then AN' is greater than AO, *Ax. a.*
and AO is greater than AM; *I.* 15
much more then is AN' greater than AM;
that is, AM is less than BN;
that is, DE is nearer to the centre than HK.

Q.E.D.

THEOR. 13. **Aliter**:

In the equal circles DEF, HKL, whose centres are A and B, let AM and BN be perpendicular to the chords DE and HK:





then according as the chord DE is greater than, equal to, or less than, the chord HK, so shall the distance AM be less than, equal to, or greater than, the distance BN.

Apply the circle HKL to the circle DEF, so that B falls on A, and BN along AM.
Let H′, N′, K′ be the new positions of H, N, K.
Then DE, H′K′ are either coincident or parallel, since they are both perpendicular to AM; *I.* 21.
therefore H′ and K′ cannot lie on opposite sides of DE.

Now if the chord DE is greater than the chord HK the minor arc DE is greater than the minor arc HK; *III.* 7.
therefore H′ and K′ are upon the minor arc DE;
therefore N′ is on the side of DE remote from A;
therefore AM is less than AN′; *Ax. a.*
that is, AM is less than BN,
and DE is nearer to the centre than HK.

Again, if the chord DE is equal to the chord HK the minor arc DE is equal to the minor arc HK; *III.* 7.
therefore H′ and K′ lie upon D and E;
therefore N′ coincides with M;

and therefore AM is equal to AN′ or BN,
and DE and HK are equally distant from the centres.

In like manner, if the chord DE is less than the chord HK, DE is further from the centre than HK.

Q.E.D.

THEOR. 14. **In the same circle, or in equal circles, chords that are equally distant from the centre are equal; and of two chords unequally distant the one nearer to the centre is the greater.**

In the equal circles DEF, HKL, whose centres are A and B, let AM and BN be respectively perpendicular to the chords DE and HK:

then according as the distance AM is less than, equal to, or greater than the distance BN, so shall the chord DE be greater than, equal to, or less than the chord HK.

For in the preceding Theorem it has been shewn that
if the chord DE is greater than the chord HK,
then AM is less than BN;
if the chord DE is equal to the chord HK,
then AM is equal to BN;
if the chord DE is less than the chord HK,
then AM is greater than BN.

Now of these hypotheses one must be true, and of the conclusions no two can be true at the same time;
hence, by the Rule of Conversion, the converse of each of the above Theorems is true.

Q.E.D.

Cor. The diameter is the greatest chord in a circle.

Ex. 28. Prove that the diameter is the greatest chord in a circle directly by *I.* 13.

Ex. 29. Prove Theor. 14 directly by *II.* 9.

EXERCISES.

30. If the straight lines which bisect at right angles the sides of a rectilineal figure all meet in a point, a circle can be described so as to pass through all the angular points of the figure.

31. Through either common point of two intersecting circles to draw the greatest straight line which has one extremity on each circumference.

32. Through either common point of two intersecting circles to draw a straight line so that the chords intercepted by the circles may be equal.

33. Through a given point within a circle to draw the shortest possible chord.

34. The chords cut off by two equal circles from a straight line parallel to the line joining their centres or passing through the middle point of it are equal.

35. A straight line ABCD cuts two concentric circles in A,B,C,D : show that AB is equal to CD.

*36. From a given point to draw the shortest and longest straight lines to the circumference of a given circle.

37. To find the shortest and longest straight lines which have one extremity on each of two given circles which do not intersect.

38. An isosceles triangle has its vertical angle equal to the exterior angle of an equilateral triangle. Prove that the radius of the circumscribing circle is equal to one of the equal sides of the given triangle.

39. Determine the centre of a given circle by means of a ruler with parallel edges whose breadth is less than the diameter of the circle.

40. From one of the points of intersection of two circles lines are drawn equally inclined to the common chord of the circles: prove that the portions of these lines intercepted between the other points in which they meet the circumferences of the circles are equal.

SECTION IV.

ANGLES IN SEGMENTS.

DEF. 9. *A* **segment** *of a circle is the figure contained by a chord and either of the arcs into which the chord divides the circumference. The segments are called* **major** *or* **minor** *segments according as the arcs that bound them are major or minor arcs.*

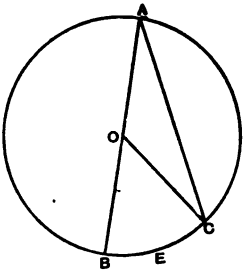
DEF. 10. *The angle formed by any two chords drawn from a point on the circumference of a circle is called an* **angle at the circumference,** *and is said to stand upon the arc between its arms.*

DEF. 11. *An angle contained by two straight lines drawn from a point in the arc of a segment to the extremities of the chord is called an* **angle in the segment.**

THEOR. 15. **An angle at the circumference is half the angle at the centre standing on the same arc.**

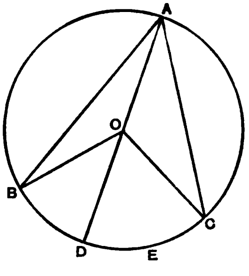
Let BAC be at an angle at the circumference, BOC an angle at the centre standing on the same arc BEC :
then shall the angle BAC be half the angle BOC.

First let the centre O lie on AB, one of the arms of the angle BAC.



Then, because OA is equal to OC, *III. Def.* 1.
therefore the angle OCA is equal to the angle OAC; *I.* 7.
therefore the angle OAC is equal to half the sum of the angles OAC, OCA;
but the angle BOC, being the exterior angle of the triangle OAC, is equal to the sum of the angles OAC, OCA; *I.* 25.
therefore the angle BAC is half the angle BOC, standing on the arc BEC.

Next, let O lie within the angle BAC.

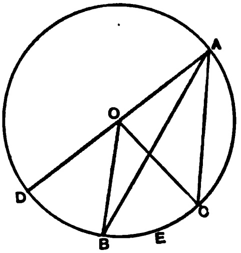


Join AO, and produce AO to meet the circumference at D.

Then, as before, the angle BAO is half the angle BOD,

and the angle OAC is half the angle DOC;
therefore the whole angle BAC is half the whole angle BOC standing on the arc BEC.

Again, let O lie without the angle BAC:



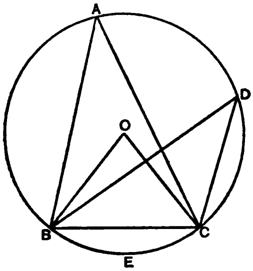
As before, join AO, and produce AO to meet the circumference at D,
then, the angle BAO being half the angle BOD, and the angle OAC half the angle DOC,
therefore the remaining angle BAC is half the remaining angle BOC. Q.E.D.

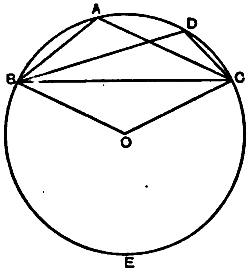
* *Ex.* 41. Two chords AB, CD intersect at a point E within the circle. Shew that the angle AEC is half the sum of the angles at the centre standing on the arcs AC, BD.

* *Ex.* 42. Two chords AB, CD, produced intersect at a point E without the circle. Shew that the angle AEC is half the difference of the angles at the centre standing on the arcs AC, BD.

THEOR. 16. **Angles in the same segment are equal to one another.**

Let BAC, BDC be angles in the same segment BADC;





then shall the angle BAC be equal to the angle BDC.

Take O the centre of the circle; join OB, OC.

Then the angles BAC, BDC are angles standing on the same arc, namely the arc BEC which is conjugate to the arc BADC, and therefore each is half of the angle BOC on the arc BEC: *III.* 15.
therefore the angle BAC is equal to the angle BDC. *Ax. h.*
Q.E.D.

Cor. 1. **The angle subtended by the chord of a segment at a point within it is greater than, and that at a point outside the segment and on the same side of the chord as the segment is less than, the angle in the segment.**

Let D be a point within the segment BAC:
then shall the angle BDC be greater than the angle in the segment BAC.

Produce BD to meet the circumference at E, and join EC.

Then, the exterior angle BDC of the triangle DEC is greater than the interior opposite angle DEC; *I.* 9.
that is, the angle BDC is greater than the angle BEC which is in the segment ABC.

In like manner it may be shown that if D is a point outside the segment ABC and on the same side of the chord BC as the segment; then the angle BDC is less than the angle in the segment BAC.

COR. 2. **The locus of a point on one side of a given finite straight line at which that line subtends a constant angle is an arc of which that line is the chord.**

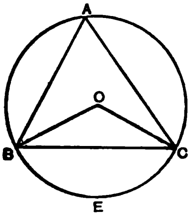
* *Ex.* 43. Two chords AB, CD of a circle intersect in E. Shew that whether E be within or without the circle the triangles AEC, DEB are equiangular. So also are the triangles EAD, ECB.

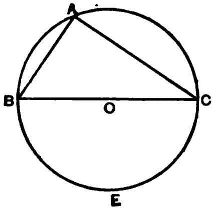
Ex. 44. BA is a chord of one, BC of the other, of two circles which intersect at B. Through B any straight line PBQ is drawn meeting the circumferences again in P and Q respectively. Shew that if PA, QC, produced if necessary, meet in R, R lies on a certain circular arc.

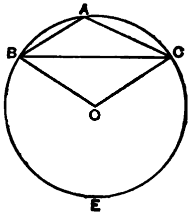
Ex. 45. P is any point on a circular arc APB, AP is produced to Q, so that PQ is equal to PB: shew that the locus of Q is a circular arc.

THEOR. 17. **The angle in a segment is greater than, equal to, or less than, a right angle, according as the segment is less than, equal to, or greater than, a semicircle.**

Let BAC be an angle in the segment BAC of the circle ABEC:







then according as the segment BAC is less than, equal to, or greater than, a semicircle;
so shall the angle BAC be greater than, equal to, or less than, a right angle.

Let O be the centre.
Join OB, OC, if necessary.

Then, according as the segment BAC is less than, equal to, or greater than, a semicircle, the arc BEC is greater than, equal to, or less than, half the circumference of the circle, and therefore the angle BOC, which stands on the arc BEC, is greater than, equal to, or less than two right angles.
But the angle BAC at the circumference is half the angle BOC at the centre on the same arc BEC; *III.* 15.
therefore according as the segment BAC is less than, equal to, or greater than, a semicircle,
the angle BAC is greater than, equal to, or less than, a right angle.
Q.E.D.

Ex. 46. The circles described on two of the sides of a triangle as diameters intersect on the base, or the base produced.

Ex. 47. The four circles described on the sides of a rhombus as diameters have one point in common.

THEOR. 18. **A segment is less than, equal to, or greater than, a semicircle, according as the angle in it is greater than, equal to, or less than a right angle.**

Let BAC be a segment of the circle ABEC, BAC an angle in the segment:
then, according as the angle BAC is greater than, equal to, or less than a right angle:
so shall the segment BAC be less than, equal to, or greater than, a semicircle.

For in the preceding Theorem it has been shewn that
if the segment BAC is less than a semicircle,
then the angle BAC is greater than a right angle;
if the segment BAC is equal to a semicircle,
then the angle BAC is equal to a right angle;
and if the segment BAC is greater than a semicircle,
then the angle BAC is less than a right angle.

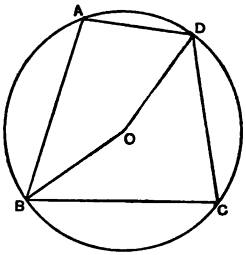
Now of these hypotheses one must be true, and of the conclusions no two can be true at the same time;
hence, by the Rule of Conversion, the converse of each of the above Theorems is true.

Q.E.D.

DEF. 12. *If the angular points of a rectilineal figure be on the circumference of a circle, the figure is said to be* **inscribed** *in the circle, and the circle to be* **circumscribed** *about the figure.*

THEOR. 19. **The opposite angles of a quadrilateral inscribed in a circle are supplementary.**

Let ABCD be a quadrilateral inscribed in the circle ABCD :



then shall the opposite angles BAD, BCD be supplementary.

Let O be the centre of the circle.
Join OB, OD.

Then the angle BAD at the circumference standing on the arc BCD is half the angle BOD at the centre standing on the same arc BCD, *III.* 15.
and the angle BCD at the circumference standing on the arc BAD is half the angle BOD at the centre standing on the same arc BAD ;
therefore the sum of the angles BAD, BCD is half the sum of the two conjugate angles of which OB, OD are the arms ;
but the sum of two conjugate angles is four right angles,
therefore the sum of the angles BAD, BCD is two right angles,
that is, the angles BAD, BCD are supplementary.

Q.E.D.

Cor. 1. **An exterior angle of a quadrilateral inscribed in a circle is equal to the interior opposite angle.**

Cor. 2. **If the opposite angles of a quadrilateral are supplementary a circle can be described about the quadrilateral.**

For if one of the angular points of the quadrilateral did not lie on the circumference of *the* circle (*III.* 12) passing through the other three, the angle at that point would not be supplementary to the opposite angle. *III.* 16, *Cor.* 1.

Ex. 48. Prove Theor. 19 by drawing the diagonals of the quadrilateral and shewing that the sum of the angles at the extremities of one diagonal is equal to the sum of the angles of either of the triangles into which the other diagonal divides the quadrilateral.

Ex. 49. If a circle can be described about a parallelogram, the parallelogram must be a rectangle.

Ex. 50. A circle can be described about every rectangle.

Ex. 51. The quadrilateral ABCD is inscribed in a circle; AB, DC produced meet in E, and BC, AD produced meet in F; prove that, if BEFD can also be inscribed in a circle, AC is a diameter of the first circle and EF of the other.

Exercises.

52. From the foot of a perpendicular AD on the hypotenuse of a right-angled triangle ABC, perpendiculars DE, DF are

drawn to the sides. Prove that B, E, F, C, are concyclic, *i.e.*, lie on the circumference of a circle.

53. AB is a chord of a circle whose centre is C, and DE is the perpendicular let fall on AB from any point D in the circumference. Prove that the angles ADE, BDC are equal.

54. AB is a fixed chord in a circle APQB; PQ another chord of given length. Shew that if AP, BQ meet in R, R will lie in the circumference of a fixed circle for all positions of PQ.

55. AOB, COD are two diameters of the circle ACBD at right angles to each other. Equal lengths OE, OF are taken along OA, OD respectively. Shew that BF produced cuts DE at right angles, and that these two straight lines, when both produced, intercept one-fourth of the circumference.

56. Through P, one of the points of intersection of two circles APB, APC, a straight line BC is drawn at right angles to AP; BA, CA meet the circles again in Q and R. Shew that AP bisects the angle QPR.

57. If two opposite sides of a quadrilateral inscribed in a circle be equal, the other two will be parallel.

58. ABCD is a parallelogram: if a circle through A,B cut AD and BC in E and F respectively, prove that another circle can be drawn through E,F,C,D. Find the position of EF when the two circles are equal.

59. The circumscribing circles of the two triangles spoken of in Book I., Theor. 20, are in all cases equal to each other.

60. The middle points of all chords of a given circle which pass through a fixed point lie on the circumference of a certain circle.

61. Through one common point of two intersecting

circles a diameter of each circle is drawn. Shew that the lines joining the other ends of the diameters pass through the other common point.

*62. P is any point on a circular arc APB. Shew that the bisector of the angle APB passes through a fixed point, and that the bisector of its supplementary angle passes through another fixed point.

63. The perpendiculars from A and C to the opposite sides of a triangle ABC intersect in E, and BD is the diameter through B of the circumscribing circle. Prove that AE is equal to CD, and that AC,ED bisect each other.

64. Shew that the circumscribing circles of the equilateral triangles described on the sides of any triangle and external to the triangle meet in a point.

65. Upon the sides of the triangle ABC the equilateral triangles A'BC, B'CA, and C'AB are constructed, the vertices A and A' being on opposite sides of BC, and so of the rest. Prove that the lines AA', BB', CC' meet in a point.

66. Prove that the points determined in Exercises 64 and 65 are the same, and if that point be D, that $AA' = BB' = CC' = DA + DB + DC$.

67. A circle is described on OA as diameter. Any line through O meets the circumference in P and the perpendicular to OA through A in Q. Shew, by Book II., Ex. 19, that the rectangle OP, OQ is constant. Also enunciate and prove the converses.

68. Squares are described on two sides of a triangle as bases, and on the third side as diagonal. Shew that the three circles about these squares have a point in common.

69. If a hexagon ABCDEF be inscribed in a circle

the sum of the angles A, C, E is equal to that of the angles B, D, F.

Enunciate and prove a similar theorem for any rectilineal figure with an even number of sides inscribed in a circle.

70. Two parallel chords AB and CD are drawn in a circle ABCD, whose centre is O. The chord CD is bisected in F, and a circle is described through A, O, F, cutting the given circle again in G. Prove that B, F, G lie in a straight line.

71. Of all triangles which have a given base and vertical angle, the greatest is that in which the two sides are equal.

72. Two circles intersect at the points A and B. In the circumference of one of the circles ABC any point P is taken, and the straight lines PA, PB (produced when necessary) meet the circumference of the other circle at the points Q, R. Shew that the chord QR will be of the same length whatever may be the position of P.

73. Through one of the points of intersection of two equal circles, each of which passes through the centre of the other, a line is drawn to intersect the circles in two other points. Prove that these points and the other point of intersection of the circles are the angular points of an equilateral triangle.

74. Through each of the points of intersection of two circles straight lines are drawn, the one meeting the circles in points A and B, and the other meeting them in points C and D. Prove that AC is parallel to BD.

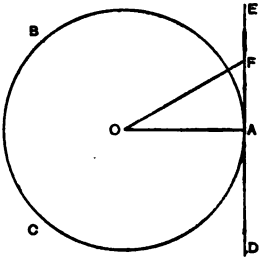
75. If two triangles have their bases, areas, and angles at the vertices equal, they are equal in all respects.

SECTION V.

TANGENTS.

THEOR. 20. **Of all straight lines passing through a point on the circumference of a circle there is one, and only one, that does not meet the circumference again, and this straight line is perpendicular to the radius to the point.**

Let A be a point on the circumference of the circle ABC whose centre is O, DAE the perpendicular through A to the radius OA:



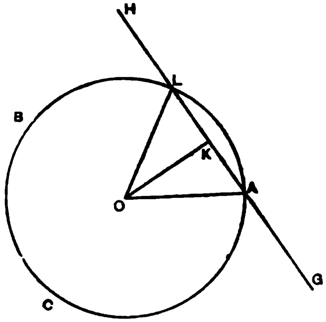
then shall A be the only point in which DE meets the circumference.

Take any point F on DE, and join OF.

Because OA is perpendicular to DE,
therefore OF is greater than OA; *I.* 15.

therefore F is without the circle ABC; *III.* 1, *Cor.*
therefore the straight line DF meets the circumference of the circle ABC in the point A only.

Again, let GAH be a straight line through A not perpendicular to OA:



then shall GH meet the circumference in a second point:

Draw OK perpendicular to GH; *I. Prob.* 3.
from O draw OL to meet GH at L, and making the angle KOL equal to the angle KOA. *I. Prob.* 5.

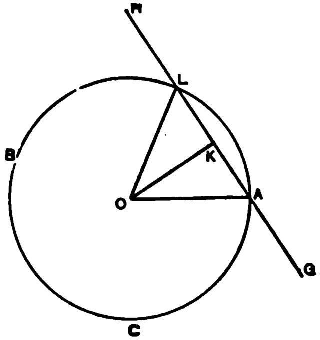
Then, because OL and OA are equally inclined to OK the perpendicular to GH,
therefore OL is equal to OA; *I.* 15.
that is, OL is equal to the radius of the circle;
therefore L is on the circumference; *III.* 1, *Cor.*
that is, the straight line GH meets the circumference in a second point L.

Q.E.D.

DEF. 13. *The straight line which, meeting the circumference of a circle in one point, does not meet it again, is said to* **touch**, *or to be a* **tangent** *to the circle at the point. The point is called the* **point of contact.**

THEOR. 20. **Aliter.**

Let A be a point on the circumference of the circle ABC whose centre is O, GAH a straight line through A not perpendicular to OA :



then shall GH meet the circumference in a second point.

Draw OK perpendicular to GH, from O draw OL to meet GH at L and making the angle KOL equal to the angle KOA.
I. Probs. 3 *and* 5.

Then, because OL and OA are equally inclined to OK the perpendicular to GH,
therefore OL is equal to OA; *I.* 15.
that is, OL is equal to the radius of the circle;

therefore L is on the circumference; *III.* 1, *Cor.*
that is, the straight line GH meets the circumference in a second point L.

Next, let the point L move along the circumference of the circle so as to continually approach to, and ultimately coincide with, the point A:
then shall GH continually approach to, and ultimately coincide with, the perpendicular through A to OA.

As the point L moves along the circumference of the circle and continually approaches to A, the minor arc AL continually diminishes,
and therefore the angle AOL continually diminishes, *III.* 5.
and when L ultimately coincides with A the angle AOL vanishes;
therefore the angle AOK, which is half the angle AOL, continually diminishes, and ultimately vanishes.
But the angle OAK is the complement of the angle AOK; *I.* 25, *Cor.*
therefore the angle OAK continually increases, and is ultimately a right angle;
that is, GH continually approaches to, and ultimately coincides with, the perpendicular through A to OA.

Q.E.D.

DEF. 13. Aliter. *If a secant of a circle alters its position in such a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to* **touch,** *or to be a* **tangent** *to, the circle. The point in which the two points of intersection ultimately coincide is called the* **point of contact.**

Theor. 20 may be restated as in the first two of the following Corollaries :—

COR. 1. **One and only one tangent can be drawn to a circle at a given point on the circumference.**

COR. 2. **Any tangent to a circle is perpendicular to the radius drawn to the point of contact.**

COR. 3. **The centre of a circle lies in the perpendicular to any tangent at the point of contact.**

For only one perpendicular can be drawn to a tangent from the point of contact.

COR. 4. **The straight line drawn from the centre perpendicular to the tangent meets it in the point of contact.**

For only one perpendicular can be drawn from the centre to the tangent.

COR. 5. PROB. **To draw a tangent to a given circle at a given point in the circumference.**

Ex. 76. To draw a tangent to a given circle making a given angle with a given straight line.

Ex. 77. All equal chords in a circle may be touched by another circle.

THEOR. 21. **A straight line cuts a circle, touches it, or does not meet it at all, according as its distance from centre is less than, equal to, or greater than, the radius.**

Let the distance from the centre be less than the radius,
then the foot of the perpendicular from the centre on the straight line is within the circumference; *III.* 1, *Cor.*
also, a point may be taken on the straight line as far from the centre as we please,
and therefore without the circumference; *III.* 1, *Cor.*
that is, there are points on the straight line both within and without the circumference;
therefore the straight line *cuts* the circumference.

Next, let the distance from the centre be equal to the radius,
then the straight line is the perpendicular to a radius at its extremity,
and therefore *touches* the circle. *III.* 20.

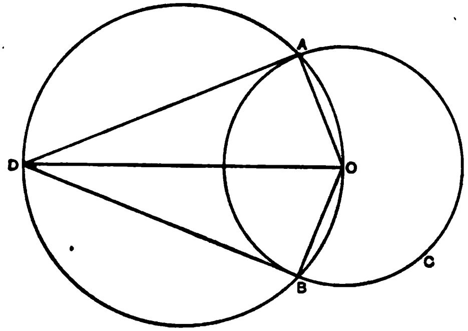
Again, let the distance from the centre be greater than the radius,
then every point on the straight line is at a distance from the centre greater than the radius, *I.* 15.
and therefore without the circumference; *III.* 1, *Cor.*
therefore the straight line does not meet the circle at all.

Q.E.D.

COR. **The distance of a straight line from the centre of a circle is less than, equal to, or greater than, the radius, according as the straight line cuts touches, or does not meet, the circle.**

THEOR. 22. **Two tangents and two only can be drawn to a circle from an external point.**

Let ABC be the given circle, D the external point:



two tangents and two only can be drawn from D to the circle ABC.

Take O the centre of the circle ABC;
join OD, and on OD as diameter describe a circle;
this circle will cut the circle ABC in two points, since O is within and D is without the circle ABC,
and in two points only. *III.* 12, *Cor.* 2.
Let A and B be these points;
join DA, DB;
DA and DB shall be the tangents drawn from D to the circle ABC.
Join OA, OB.

The angle DAO in the semicircle DAO is a right angle; *III.* 17.
therefore DA touches the circle ABC. *III.* 20
In like manner BD touches the circle.

Again, any other straight line drawn from D to meet the circle ABC, as DE, is not a tangent to ABC, since the angle DEO is greater than, or less than, a right angle, according as E is within, or without the circle ABD. *III.* 16, *Cor.* 1.

Q.E.D.

COR. 1. **The two tangents drawn to a circle from an external point are equal, and make equal angles with the straight line joining that point and the centre.**

I. 20, *Cor.* 1.

COR. 2. PROB. **To draw a tangent to a given circle from a given point outside the circumference.**

The construction and demonstration are contained in the Theorem.

* *Ex.* 78. If a circle touch each of two intersecting straight lines, its centre lies on one or other of the bisectors of the angles contained by these lines.

* *Ex.* 79. If a circle touch each of two parallel straight lines, its centre lies on the line parallel to both and midway between them.

Ex. 80. A circle is inscribed in a right-angled triangle. Shew that the hypotenuse together with the diameter is equal to the sum of the other two sides.

* *Ex.* 81. If a quadrilateral be circumscribed about a circle the sum of two opposite sides is equal to that of the other two opposite sides.

* *Ex.* 82. If a convex quadrilateral be such that the sum of

one pair of opposite sides is equal to that of the other pair, a circle may be inscribed in it.

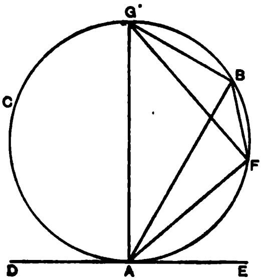
Ex. 83. If a rectilineal figure of any even number of sides be described about a circle, the sum of one set of alternate sides is equal to that of the other set.

Ex. 84. To draw in a given circle a chord of given length which shall pass through a given point.

THEOR. 23. **The adjacent angles contained by a tangent and a chord drawn from the point of contact are equal to the angles in the segments into which the chord divides the circle, each to the angle in the segment alternate to it.**

First Proof.

Let DE be the tangent to the circle ABC at the point A, AB a chord drawn from A:



then shall the acute angle BAE be equal to the angle in the segment ACB, alternate to BAE.

Draw the diameter AG; join BG.

Because DE touches the circle at A, and AG is a diameter, therefore GAE is a right angle; *III.* 20.
therefore the angle BAE is the complement of the angle BAG. *I.* 25, *Cor.*

Again, because ABG is a semicircle,
therefore the angle ABG is a right angle; *III.* 17.
therefore the angle BGA is also the complement of the angle BAG; *I.* 25, *Cor.*
therefore the angle BAE is equal to the angle BGA, *I.* 1, *Cor.* 2.
that is, to an angle in the segment ACB.

Also the obtuse angle BAD shall be equal to the angle in the segment AFB, alternate to BAD.

Join FA, FB, FG.

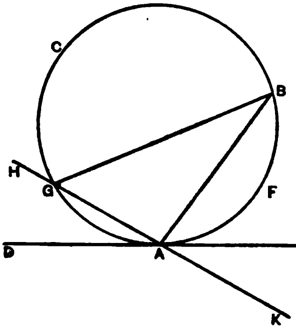
Then the angle GAD is equal to the angle GFA, since each is a right angle, and the angle GAB is equal to the angle GFB, since they are in the same segment;
therefore the whole angle BAD is equal to the whole angle AFB; that is, to an angle in the segment AFB.

Q.E.D.

Ex. 85. Apply Theor. 19 to prove the second part of Theor. 23.

Second Proof.

Let DE be the tangent to the circle ABC at the point A, AB a chord drawn from A:



then shall the angle BAE be equal to the angle in the segment ACB alternate to BAE, and the angle BAD to the angle in the segment AFB alternate to BAD.

Take any point G in the arc ACB, and draw the secant HGAK; join GB.

The angle BGK remains constant in magnitude whatever be the position of G in the arc ACB. *III.* 16.

Let G move up to, and ultimately coincide with, A:
then the secant HK ultimately becomes the tangent at A, *III. Def.* 13, *Aliter.*
and the angle BGK ultimately becomes the angle BAE;
therefore the angle BAE is equal to the angle in the segment ACB, alternate to BAE.

In like manner it may be shewn that the angle BAD is equal to the angle in the segment AFB, alternate to BAD.

Q.E.D.

Ex. 86. Prove Theor. 23 as a limiting case of Theor. 19, Cor. 1, two of the vertices of the quadrilateral being made to coincide.

Ex. 87. AB, AC are two chords of a circle; BD is drawn parallel to the tangent at A to meet AC in D; prove that the circle BCD touches AB.

Ex. 88. ABCD is a quadrilateral inscribed in a circle, and its diagonals intersect in E. About the triangle AEB a circle is described. Prove that the straight line touching this circle at the point E is parallel to one of the sides of the figure ABCD.

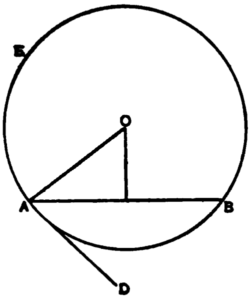
Ex. 89. Any chord of a circle bisects the angle between the diameter through one extremity and the perpendicular from it on the tangent at the other.

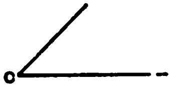
Ex. 90. In a right-angled triangle, if a circle be described on one of the sides containing the right angle as diameter, the tangent at the point where it cuts the hypotenuse bisects the other side.

Ex. 91. Draw a tangent at a given point on a given circle without first finding the centre.

PROB. 3. **On a given straight line to describe a segment of a circle containing a given angle.**

Let AB be the given straight line, C the given angle:
it is required to describe on AB a segment of a circle containing an angle equal to C.





Make the angle BAD equal to the given angle C; *I. Prob.* 5.
from A draw a straight line perpendicular to AD; *I. Prob.* 2.
draw a straight line bisecting AB at right angles; *I. Prob.* 4.
let these two straight lines meet at O;
with centre O and radius OA describe the circle ABE:
AEB, the segment alternate to BAD, shall be the segment required.

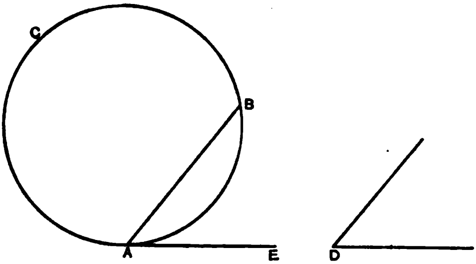
Because the straight line bisecting AB at right angles is the locus of points equidistant from A and B, *Loci, iii.*
therefore O is equidistant from A and B;
therefore B lies on the circumference of the circle whose centre is O and radius OA. *III.* 1, *Cor.*

Again, because the angle OAD is a right angle,
therefore AD is a tangent to the circle; *III.* 20.
therefore the angle BAD is equal to the angle in the alternate segment AEB; *III.* 23.
but the angle BAD is equal to the angle C; *Constr.*
therefore the angle in the segment AEB is equal to the given angle C;
therefore AEB is the segment required. Q.E.F.

Ex. 92. Examine the case in which the given angle is a right angle.

PROB. 4. **From a given circle to cut off a segment containing a given angle.**

Let ABC be the given circle, D the given angle:
it is required to cut off from ABC a segment containing an angle equal to the angle D.



At any point A on the circumference of the circle draw the tangent AE, *III.* 20, *Cor.* 5.
and draw the chord AB, making with AE an angle equal to D: *I. Prob.* 5.
ACB, the segment alternate to BAE, shall be the segment required.

Because AE is a tangent to the circle, and AB a chord drawn from the point of contact,
therefore the angle BAE is equal to the angle in the segment ACB, alternate to BAE. *III.* 23.

But the angle BAE is equal to the angle D ; *Constr.*
therefore the angle in the segment ACB is equal to the given angle D.

Q.E.F.

EXERCISES.

93. Two equal circles have their centres at A and B, O is a fixed point outside those circles, A is the centre of a third circle whose radius is equal to OB ; prove that the tangents from O to the three circles are equal to the sides of a right-angled triangle.

94. Describe a triangle having given the vertical angle, one of the sides containing it, and the altitude.

95. Draw a circle through a given point to touch a given straight line at a given point.

96. Tangents are drawn to the circumscribing circle of an isosceles triangle through the angular points. Shew that they form an isosceles triangle. Shew also that the two triangles cannot have equal vertical angles unless they are both equilateral.

97. If a parallelogram be inscribed in, or circumscribed about a circle, its diagonals pass through the centre.

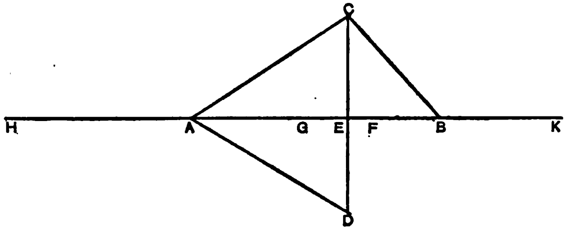
SECTION VI.

TWO CIRCLES.

DEF. 14. *Two circles are said to* **touch externally** *at a point where they meet, if each lies outside the other; to* **intersect** *if a part of each lies inside, and the remaining part outside the other; and to* **touch internally** *at a point where they meet, if one of them lies inside the other.*

THEOR. 24. **If two circles meet in a point which is not on the line joining their centres, they meet in one other point; and the circles intersect; the line joining the two points of intersection is bisected at right angles by the line joining their centres; and the distance between the centres is greater than the difference, and less than the sum, of the radii.**

Let the circumference of two circles whose centres are A and B meet in the point C, not on the straight line through A and B:



then shall they meet in one other point.

Join AB.

Draw CE perpendicular to AB, and produce CE to D, so that ED may be equal to CE. *I. Prob.* 3.

In the triangles AEC, AED,
the side CE is equal to the side DE,
the side AE is common to both,
and the angle AEC is equal to the angle AED ;
therefore the side AC is equal to the side AD ; *I.* 5.
therefore D is a point on the circumference of the circle whose centre is A. *III.* 1, *Cor.*

In like manner, D is a point on the circumference of the circle whose centre is B.
Therefore the circles meet in a second point D, and they cannot meet again. *III.* 12, *Cor.* 2.

Also the circles shall intersect.

Let the line through A and B cut the circle whose centre is A in F and H, and the other circle in G and K, F being on the same side of A as B, and G on the same side of B as A.

Because BK is equal to BC,
therefore AK is equal to the sum of AB and BC ;
therefore AK is greater than AC ; *I.* 13.
therefore K is outside the circle whose centre is A. *III.* 1, *Cor.*

Again, because BG is equal to BC ;
therefore AG is equal to the difference of AB and BC. *Ax. e.*
But the difference of AB and BC is less than the third side AC of the triangle ABC,

therefore AG is less than AC; *I.* 13, *Cor.*
therefore G is inside the circle whose centre is A. *III.* 1, *Cor.*
Hence the circles intersect. *III. Def.* 14.
Again, the line AB bisects the line CD at right angles, by construction;
and AB is less than the sum of AC and BC, *I.* 13.
and greater than the difference of AC and BC. *I.* 13, *Cor.*

Q.E.D.

COR. **If two circles meet in one point only, that point lies on the line joining the centres.**

For it has been shown that if the circles have a common point not on the line joining their centres, then they must meet in more than one point
hence, by contraposition, if the circles meet in one point only, that point lies on the line joining their centres.

THEOR. 25. **If two circles meet in a point which is on the line joining their centres they do not meet in any other point, but touch either externally or internally; and the distance between their centres is in the former case equal to the sum, and in the latter to the difference, of the radii.**

Let the circumferences of two circles whose centres are A and B meet at the point C on the straight line through A and B:

Fig. 1.

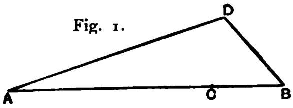
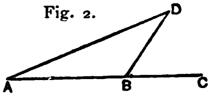


Fig. 2.



they shall not meet in any other point.

Take any other point D on the circumference of the circle whose centre is B; join AD and BD.

Then in fig. 1 the sum of AD and BD is greater than AB, *I.* 13.
and BD is equal to BC;
therefore AD is greater than AC; *Ax. g.*
therefore D is outside the circle whose centre is A; *III.* 1, *Cor.*
therefore the circle whose centre is B is wholly outside the circle whose centre is A;
therefore the circles touch externally. *III. Def.* 14.

In fig. 2 AD is less than the sum of AB and BD, *I.* 13.
and BD is equal to BC;
therefore AD is less than AC;
therefore D is within the circle whose centre is A; *III.* 1, *Cor.*
therefore the circle whose centre is B is wholly within the circle whose centre is A;
therefore the circles touch internally. *III. Def.* 14.

Also in fig. 1 AB is equal to the sum, and in fig. 2 to the difference, of the radii AC and BC.

Q.E.D.

COR. 1. Hence by contraposition, **if two circles intersect, neither point is on the line joining their centres.**

COR. 2. The Cor. to Theor. 24 may also be stated thus:

If two circles touch one another (whether internally or externally) the line joining their centres passes through the point of contact.

COR. 3. **If two circles touch one another, they have a common tangent at the point of contact.**

For the straight line drawn from the point of contact perpendicular to the radius of one circle will be perpendicular to the radius of the other, and will therefore be a tangent to both circles. *III.* 20, *Cor.* 2.

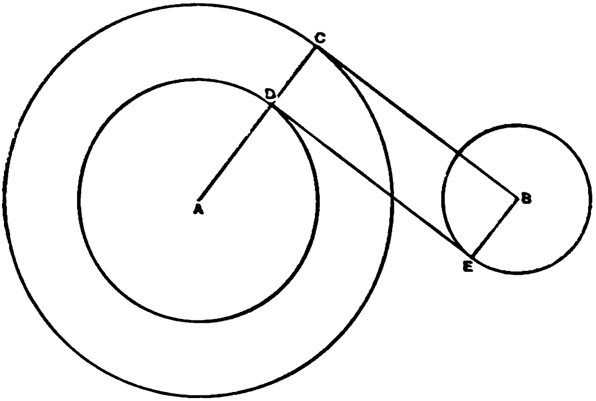
**Ex.* 98. If two circles do not meet and lie each wholly outside the other, the distance between their centres is greater than the sum of their radii; and if one lies wholly inside the other, that distance is less than the difference of their radii.

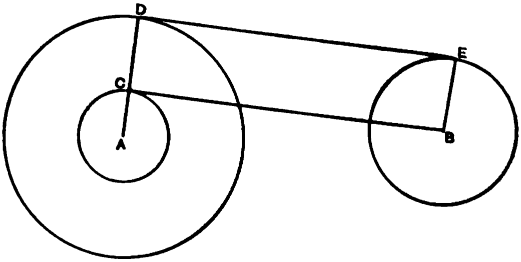
**Ex.* 99. According as the distance between the centres of two circles is (1) greater than the sum of the radii, (2) equal to that sum, (3) less than the sum but greater than the difference of the radii, (4) equal to the difference of the radii, (5) less than the difference; so do the circles (1) lie each wholly outside the other, (2) touch each other externally, (3) intersect, (4) touch each other internally, (5) lie one inside the other.

Shew that these statements follow from the preceding statements by the Rule of Conversion, and give also a direct geometrical proof of each.

PROB. 6. **To draw a common tangent to two given circles.**

Let A and B be the centres of the given circles of which the former is not the less:
it is required to draw a common tangent to the circles,





With centre A, and radius equal to the sum or difference of the radii of the given circles, describe a circle; *Post.* 3.
from B draw BC to touch this circle at C; *III.* 22, *Cor.* 2.
join AC, and let AC, or AC produced through C, meet the circumference of the circle whose centre is A at D;
through B draw BE parallel to CD, and on the same side of BC as CD, to meet the circumference of the circle whose centre is B at E. *I. Prob.* 6.

Join DE:
then shall DE be a common tangent to the given circles.

Because BC is a tangent to the circle through C,
therefore the angle ACB is a right angle. *III.* 20.

Again, AC is equal to the sum or difference of AD and BE;
therefore BE is equal to CD,
and BE is parallel to CD;
therefore BCDE is a parallelogram; *I.* 31.
and BCD one of its angles is a right angle;
therefore the angles CDE and BED are also right angles;
I. 28, *Cor.*
therefore DE touches both the given circles. *III.* 20.
Q.E.F.

If the given circles are wholly outside each other, AB is greater than the sum of their radii, and B lies outside each of the circles used in the construction, and four common tangents can be drawn.

If the circles touch each other externally, AB is equal to the sum of their radii, and B lies on the circumference of one of the circles used in the construction and outside the other, and three common tangents can be drawn;

if the circles intersect, AB is less than the sum and greater than the difference of their radii, and B lies within one and outside the other of the circles used in the construction, and two common tangents can be drawn.

If the circles touch internally, AB is equal to the difference of their radii, and B lies within one, and on the circumference of the other of the circles used in the construction, and one common tangent can be drawn.

If one of the circles is wholly within the other, AB is less than the difference of their radii, and B lies within each of the circles used in the construction, and no common tangent can be drawn.

Ex. 100. Give the construction of this Problem when the circles are equal.

Exercises.

101. Draw a straight line whose extremities and points of trisection lie on the circumferences of two equal circles which touch each other.

102. Two given circles touch each other externally at P and are touched by the straight line AB at A and B respectively; shew that the circle on AB as diameter passes through P and touches the line joining the centres of the given circles.

103. If three equal circles touch each other, their centres are the corners of an equilateral triangle. So also are the points of contact.

104. If two circles touch externally at E, and AB, CD be any two parallel diameters of the circles, shew that the straight lines AD, BC will pass through E.

105. If the radius of one circle is the diameter of another, any straight line drawn from the point of contact to the outer circumference is bisected by the interior one.

106. Draw a circle of given radius to touch a given circle at a given point. When will there be only one solution?

107. Draw a circle with its centre at a given point to touch a

given circle. Shew that there are generally two solutions. When will there be only one?

108. Two circles touch externally. Shew how to place a line of given length so that it shall pass through the point of contact and have its extremities on the circumferences of the circles.

109. Two equal circles have a common chord AB. If a chord AC of one of them, equal to AB, when produced backwards passes through the centre of the other: shew that AB is equal to the radius of either circle.

110. Two circles touch externally. Shew that the square on the common tangent is equal to the rectangle contained by their diameters.

111. Given two circles and a point A on the circumference of one of them, draw a chord PA so that if it be produced to cut the other circle in Q, QA may be equal to PA.

SECTION VII.

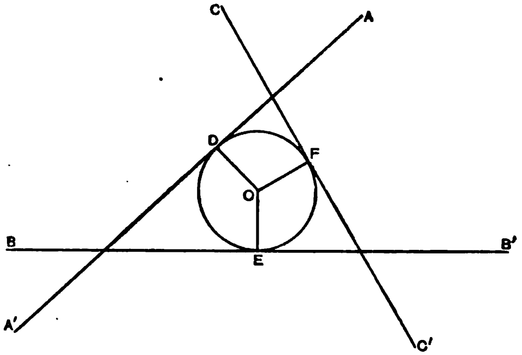
INSCRIBED AND CIRCUMSCRIBED FIGURES.

PROB. 6. **To describe a circle passing through three given points which are not in the same straight line.**

The construction and proof are contained in Theorem 12.

Prob. 7. **To describe the circles touching three given straight lines which intersect one another, but not in the same point.**

Let AA′, BB′, CC′ be three given straight lines which intersect but not in the same point:
it is required to describe a circle touching AA′, BB′, and CC′.



Find the four points equidistant from AA′, BB′, and CC′.
Inter. of Loci, ii.

Let O be any one of these points:
draw OD perpendicular to AA′, and describe a circle with centre O and radius OD:
this shall be a circle touching the given lines.

Draw OE and OF perpendicular to BB′, and CC′;
then, because O is equidistant from AA′, BB′, and CC′,
therefore OE and OF are each equal to OD;
therefore E and F lie on the circumference of the circle whose centre is O and radius OD, *III.* 1. *Cor.*

and the circle touches each of the lines AA′, BB′, CC′, since the angles at D, E, F are right angles. *III.* 20.
Similarly with each of the other points as centre a circle may be drawn.
Therefore four circles touching AA′, BB′, and CC′ have been drawn.

Q.E.F.

COR. **If two parallel straight lines are intersected by a third straight line, two and only two circles can be drawn touching the three straight lines.**

DEF. 15. *The circle that touches the three sides of a triangle is called the* **inscribed** *circle of the triangle.*

DEF. 16. *A circle that touches one side of a triangle and the other two sides produced is called an* **escribed** *circle of the triangle.*

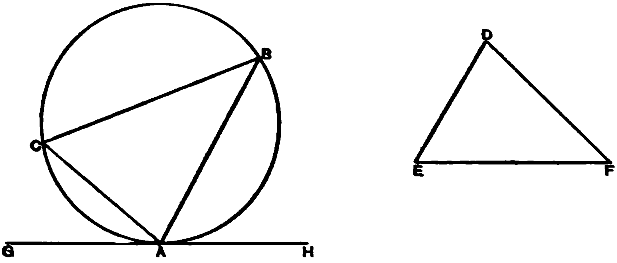
Ex. 112. If two parallel straight lines are intersected by a third straight line, shew that the circles touching the three straight lines are equal.

**Ex.* 113. The line joining the centres of two escribed circles passes through an angular point of the triangle and is perpendicular to the line joining the centre of the inscribed to that of the third escribed circle.

Ex. 114. The radii of the two equal escribed circles of an isosceles triangle are equal to the altitude of the triangle.

PROB. 8. **In a given circle to inscribe a triangle equiangular to a given triangle.**

Let ABC be the given circle, DEF the given triangle: it is required to inscribe in ABC a triangle equiangular to DEF.



At any point A on the circumference of ABC draw the tangent GH. *III.* 20, *Cor.* 5.
From A draw the chord AB making the angle HAB equal to the angle DEF, *I. Prob.* 5.
and the chord AC making the angle GAC equal to the angle DFE.
Join BC:
then shall ABC be the triangle required.

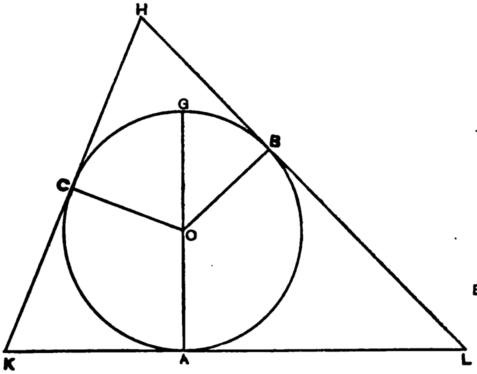
The angle HAB is equal to the angle ACB in the alternate segment of the circle. *III.* 23.
But the angle DEF is equal to the angle HAB;
therefore the angle ACB is equal to the angle DEF. *Ax. c.*
In like manner the angle ABC is equal to the angle DFE;
therefore the remaining angle BAC is equal to the remaining angle EDF; *I.* 25.
therefore the triangle ABC is equiangular to the triangle DEF, and it is inscribed in the circle ABC.

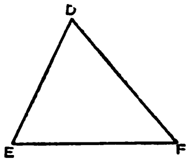
Q.E.F.

DEF. 17. *If all the sides of a rectilineal figure touch a circle lying within the figure, the circle is said to be* **inscribed** *in the figure, and the figure to be* **circumscribed** *about the circle.*

PROB. 9. **About a given circle to circumscribe a triangle equiangular to a given triangle.**

Let ABC be the given circle, DEF the given triangle:
it is required to circumscribe about ABC a triangle equiangular to DEF.





Find the centre O, and draw any diameter AG; make the angle GOC equal to the angle DEF, and on the other side of AG make the angle GOB equal to the angle DFE;

III. Prob. 2, *and I. Prob.* 5.

draw tangents to the circle at the points A, B, C, *III.* 20, *Cor.* 5.
forming the triangle HKL:
HKL shall be the triangle required.

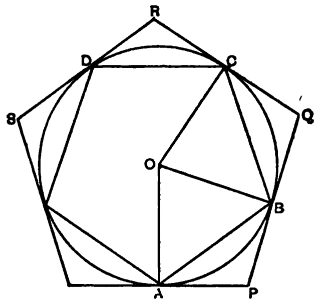
Because the angles at A and C are right angles, *III.* 20, *Cor.* 2.
therefore a circle may be described about the quadrilateral CKAO; *III.* 19, *Cor.* 2.
therefore the angle CKA is equal to the exterior angle COG, *III.* 19, *Cor.* 1.
and therefore to the angle DEF. *Constr.*
In like manner the angle ALB is equal to the angle DFE;
hence the remaining angle KHL is equal to the remaining angle EDF, *I.* 25.
and the triangle HKL is equiangular to the triangle DEF,
and it is circumscribed about the circle ABC.

Q.E.F.

Ex. 115. About a given circle to circumscribe a quadrilateral equiangular to a given quadrilateral.

THEOR. 26. **If the whole circumference of a circle is divided into any number of equal arcs, the inscribed polygon formed by the chords of these arcs is regular; and the circumscribed polygon formed by tangents drawn at all the points of division is also regular.**

Let the whole circumference of the circle ABC be divided into any number (say five as in the figure) of equal arcs at the points A, B, C . . . , and let the chords AB, BC, CD, . . . be drawn:



then shall the inscribed polygon ABCD . . . be regular.

Because the arcs AB, BC, CD, . . . are all equal, *Hyp.*
therefore the chords AB, BC, CD, . . . are all equal; *III.* 6.
therefore the polygon ABCD . . . is equilateral.

Again, each angle of the polygon ABCD . . . stands upon an arc which is less than the whole circumference by two of the arcs AB, BC, CD . . ., but the arcs AB, BC, CD, . . . are all equal; *Hyp.*
therefore the remaining arcs on which the angles of the polygon stand are all equal; *Ax. e.*
therefore the polygon is equiangular. *III.* 5 *and* 15.
Hence the polygon is regular. *I. Def.* 23.

Next, let tangents be drawn at the points A, B, C, D, . . . so as to form a polygon PQRS . . . .: *III.* 20, *Cor.* 5.
then shall the circumscribed polygon PQRS . . . be regular.

Find the centre O, and join OA, OB, OC. *III. Prob.* 2.
Rotate the figure OAPB round O until A coincides with B:
then because the arc AB is equal to the arc BC, *Hyp.*

therefore B will fall on C,
and the tangents AP and BP will coincide with the tangents BQ and CQ, since each tangent is at right angles to the radius drawn to its point of contact; *III.* 20, *Cor.* 2.
therefore the figure OAPB coincides with the figure OBQC;
therefore AP is equal to BQ.
But the tangents AP and BP are equal, *III.* 22, *Cor.* 1.
therefore the tangents AP, BP, BQ, CQ . . . are all equal,
and therefore the polygon PQRS . . . is equilateral, since each of its sides is made up of two of these tangents.

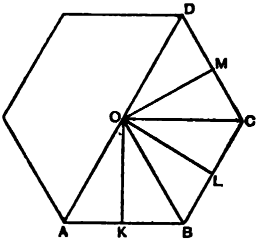
Again, the angle APB coincides with the angle BQC, and is therefore equal to it;
that is, any angle of the polygon is equal to the next succeeding angle;
therefore the polygon is equiangular.
Hence the polygon is regular. *I. Def.* 23.
Q.E.D.

Ex. 116. Any equilateral figure inscribed in a circle is also equiangular.

Ex. 117. Any equiangular figure inscribed in a circle is also equilateral.

THEOR. 27. **The bisectors of the angles of a regular polygon meet in a point which is equidistant from all the vertices of the polygon and from all the sides.**

Let ABCD . . . be a regular polygon, and let the angles of the polygon at A and B be bisected by straight lines meeting at O; join OC, OD . . . :



then shall OC, OD . . . bisect the angles of the polygon at C, D . . .

In the triangles OBA, OBC,
the side AB is equal to the side BC, *Hyp.*
the side OB is common to both,
and the angle OBA is equal to the angle OBC; *Hyp.*
therefore the angle OAB is equal to the angle OCB. *I.* 5.
But the angle OAB is half the angle of the polygon at A, *Hyp.*
and the angle at A is equal to the angle at C; *Hyp.*
therefore the angle OCB is half the angle at C,
that is, OC bisects the angle at C.
In like manner OD . . . bisects the other angles of the polygon.

Again, because the angles of the polygon at A and B are equal, *Hyp.*
therefore the angles OAB, OBA, the halves of these, are also equal; *Ax. h.*
therefore OA is equal to OB. *I.* 7.
In like manner OB is equal to OC, and so on.
Hence the lines OA, OB, OC, OD . . . are all equal.

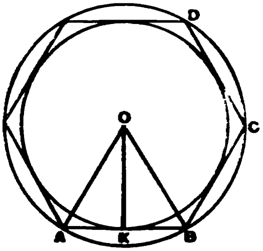
Again, draw OK, OL, OM . . . perpendicular to the sides

AB, BC, CD, . . . : *I. Prob.* 3.
then shall OK, OL, OM . . . be all equal.

In the triangles OBK, OBL,
the angle OKB is equal to the angle OLB,
the angle OBK is equal to the angle OBL, *Hyp.*
and the side OB is common to both;
therefore OK is equal to OL. *I.* 19.
In like manner OL is equal to OM, and so on.
Hence the perpendiculars OK, OL, OM . . . are all equal.
Q.E.D.

PROB. 10. **To circumscribe a circle about, or to inscribe a circle in, a given regular figure.**

Let ABCD . . . be the given regular polygon: it is required to circumscribe a circle about, and to inscribe a circle in, the polygon ABCD . . .



Bisect two adjacent angles A, B of the polygon, and let the
bisectors meet at O, *I. Prob.* 1.
draw OK perpendicular to AB, *I. Prob.* 3.
describe circles with centre O, and radii OA [illegible]

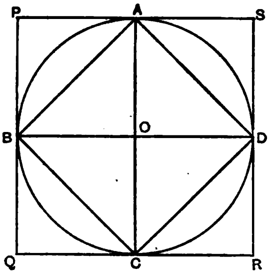
these circles shall be the circumscribed and inscribed circles required.

Because OA, OB, OC . . . are all equal, *III.* 27.
therefore B, C . . . lie on the circumference of the circle whose centre is O and radius OA; *III.* 1, *Cor.*
therefore this circle is the circumscribed circle of the polygon ABCD . . .

Again, all the perpendiculars from O on the sides of the polygon are equal; *III.* 27.
therefore the feet of these perpendiculars lie on the circumference of the circle whose centre is O and radius OK,
and each of the sides will touch this circle; *III.* 20.
therefore the circle is the inscribed circle of the polygon ABCD . . .
Q.E.F.

PROB. 11. **To inscribe in, or to circumscribe about, a given circle regular figures of 4, 8, 16, 32 . . . sides.**

Let ABCD be the given circle: it is required to inscribe in, or to circumscribe about, ABCD regular figures of 4, 8, 16, 32 . . . sides.



Find the centre O, and draw two diameters AC, BD, at right angles to each other. *I. Prob.* 2.
Join AB, BC, CD, DA,
and draw tangents to the circle at A, B, C, D, so as to form the figure PQRS: *III.* 20, *Cor.* 5.
ABCD and PQRS shall be the inscribed and circumscribed regular figures of four sides.

The angles at O are all right angles; *Constr.*
therefore the arcs AB, BC, CD, DA are all equal; *III.* 4.
therefore the figures ABCD, PQRS are regular. *III.* 26.

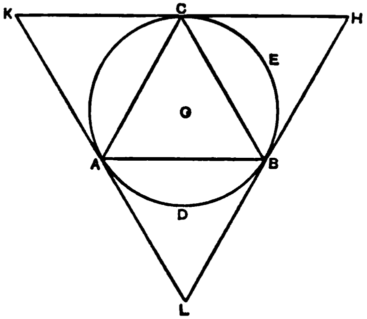
To inscribe, and circumscribe, regular figures of eight sides, bisect the arcs AB, BC, CD, DA, *III. Prob.* 1.
and proceed as before.

By repeating this process figures of 16, 32 . . . sides may be drawn.

Q.E.F.

PROB. 12. **To inscribe in, or to circumscribe about, a given circle, regular figures of 3, 6, 12, 24 . . . sides.**

Let ABC be the given circle: it is required to inscribe in, or to circumscribe about, ABC regular figures of 3, 6, 12, 24 . . . sides.



Find the centre O. *III. Prob.* 2.

With any point D on the circumference of the given circle as centre, and with radius equal to the radius of the given circle, describe a circle cutting the circumference ABC at A and B; with centre B and the same radius describe a circle cutting ABC at E, and with centre E and the same radius describe a circle cutting ABC at C.

Join AB, BC, CA,

and draw tangents at ABC so as to form the figure HKL:

ABC and HKL shall be the required inscribed and circumscribed regular figures of 3 sides.

Because each of the triangles AOD, DOB, BOE, EOC is equilateral, *Constr.*

therefore each of the angles AOD, DOB, BOE, EOC is a third of two right angles; *I.* 25.

therefore each of the angles AOB, BOC is a third of four right angles.

But all the angles round O are together equal to four right angles; therefore the remaining angle COA is also a third of four right angles;

therefore the arcs AB, BC, CA are all equal; *III.* 4.
therefore the figures ABC, HKL are regular. *III.* 26.

Figures of 6, 12, 24 . . . sides may be obtained by repeatedly bisecting the arcs AB, BC, CA, and drawing chords and tangents as before. Q E.F.

Ex. 118. To describe a regular hexagon ABCDEF, such that the diagonal AC may be of given length.

EXERCISES.

119. Two circles are described, each touching one side of a triangle, and the other two sides produced. Shew that a circle can be described passing through their centres and two of the angular points of the triangle.

*120. If ABC be a triangle, shew that the circle through B, C and the centre of the escribed circle touching BC passes through the centre of the inscribed circle.

121. Through each angular point of any triangle two straight lines are drawn parallel to the lines joining the centre of the circumscribed circle to the other angles of the triangle: prove that these straight lines will form an equilateral hexagon, and that each of the angles of this hexagon is equal to one of its other angles, and double one of the angles of the triangle.

122. The radii of the inscribed, circumscribed, and escribed circles of an equilateral triangle are in the proportion of 1, 2, 3.

123. The equilateral triangle, square, and regular hexagon inscribed in a given circle are respectively one-fourth, one-half, and three-fourths of the corresponding figures described about it.

124. Find the angles subtended at the centre of a circle by the three segments made by the sides of the circumscribed square on any tangent to the circle.

125. With three given points as centres to describe three circles touching each other. Shew that there are, in general, four solutious.

126. If the straight lines which bisect the angles of a rectilineal figure all pass through one point, a circle may be inscribed in the figure.

127. Shew that a regular polygon has, or has not, a centre of symmetry, according as the number of its sides is even or odd.

128. A regular polygon of n sides has n or $2n$ axes of symmetry, according as n is odd or even.

129. Describe a circle to touch each of two given lines, and having its centre at a given distance from a third given line. How many such circles can be described?

130. Find a point O within a given triangle such that, if AO, BO, CO be joined, the angles OBC, OCA, OAB shall all be equal to one another.

131. If the centres of the inscribed and circumscribed circles of a triangle coincide, the triangle must be equilateral.

132. A circle is inscribed in an isosceles triangle. Shew that the triangle formed by joining the points of contact is also isosceles.

133. The locus of the centre of the inscribed circle of a triangle, whose base and vertical angle are given, consists of two circular arcs.

134. If DA be one side of a regular hexagon in a circle, AB a tangent equal to it and making an obtuse angle with

it, C the centre of the circle, and if BD meet the circle in E and BC meet the nearer part of the circumference in F, prove that AE and EF are equal to sides of regular figures of twelve and twenty-four sides respectively inscribed in the same circle.

135. Construct an isosceles triangle having given its base equal to the greater, and the diameter of its inscribed circle equal to the less, of two given straight lines.

136. In a given right-angled triangle the sides containing the right angle are 6 and 8 respectively. Find the lengths of the segments into which the hypotenuse is divided by the inscribed circle.

137. Construct a triangle having given the vertical angle and the segments of the base made by the inscribed circle.

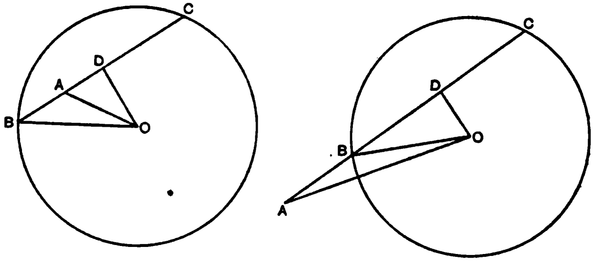
138. A square and an equilateral triangle are inscribed in the same circle. Prove that the area of the square is two-thirds of the square on the side of the triangle.

SECTION VIII.

THE CIRCLE IN CONNECTION WITH AREAS.

THEOR. 28. **If a chord of a circle is divided into two segments by a point in the chord or in the chord produced, the rectangle contained by these segments is equal to the difference of the squares on the radius and on the line joining the given point with the centre of the circle.**

Let BC a chord of a circle whose centre is O be divided into two segments at the point A:



then shall the rectangle contained by AB and AC be equal to the difference of the squares on OB and OA.

Draw OD perpendicular to BC;
then BD is equal to DC. *III.* 9.
Hence AC is equal to the sum, and AB to the difference, of BD and AD;
therefore the rectangle contained by AB and AC is equal to the difference of the squares on BD and AD. *II.* 8.

But the square on OB is equal to the sum of the squares on BD and OD, *II.* 9.
and the square on OA is equal to the sum of the squares on AD and OD; *II.* 9.
therefore the difference of the squares on OB and OA is equal to the difference of the squares on BD and AD. *Ax. e.*
Hence the rectangle contained by AB and AC is equal to the difference of the squares on OB and OA. *Ax. c.*

Q.E.D.

COR. 1. **The rectangle contained by the segments of any chord passing through a given point is the same whatever be the direction of the chord.**

COR. 2. **If the point is within the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on half that chord which is bisected by the given point.** [Particular case of Cor. 1.]

COR. 3. **If the point is without the circle, the rectangle contained by the segments of any chord passing through it is equal to the square on the tangent to the circle drawn from that point.** [Particular case of Cor. 1.]

COR. 4. **Conversely, if the rectangle contained by the segments of a chord passing through an external point is equal to the square on the line joining that point to a point in the circumference of the circle, this line touches the circle.**

For, if the rectangle contained by AB, AC is equal to the square on AP, and AP does not touch the circle, AP will meet the circle again in Q; therefore, by Cor. 1, the rectangle contained by AP, AQ is equal to the rectangle contained by AB, AC, which by hypothesis is equal to the square on AP; but this is impossible; therefore AP touches the circle at P.

Ex. 139. Conversely to Cor. 2, shew that if the rectangle contained by the segments of a chord passing through a point within the circle is equal to the square on a line joining that point to a point in the circumference of the circle, this line produced through the point to meet the circumference will be bisected at the point.

Ex. 140. A circular arch of 60 feet span is 18 feet high: find the radius of curvature of the arch.

Ex. 141. If the water flowing through this arch were to rise 14 feet, by how much would the span be diminished?

Ex. 142. An arch is to be built so as to give a roadway 60 feet wide and a pathway on each side 10 feet wide with a height at the curb of the path of 10 feet; find the radius of curvature and height of the arch.

Ex. 143. Assuming that an eye six feet above the level of a still lake can *just* see a light also six feet above the level of the lake at a distance of six miles: find the diameter of the earth in miles.

Ex. 144. From the data of the last question, shew that if h be the height of the eye above the sea in feet, and d the distance of the offing in miles, $d = \sqrt{\frac{3}{2}h}$.

Ex. 145. The lantern of a lighthouse is 96 feet above the sea level, to what distance does it illuminate the sea? and how far would it be visible to an eye 24 feet above the surface?

Ex. 146. What is the extreme distance at which two mountain summits, each 1,350 feet high, would be visible to one another?

Ex. 147. Shew why a ship sailing away from the land disappears more and more rapidly from a spectator on the land, as she gets farther away.

Ex. 148. Compare the diameters of two globes, on one of which objects of a given height are just visible to one another at double the extreme distance at which they are so on the other.

Ex. 149. On a railway on a level prairie a water tank is raised on columns $13\frac{1}{2}$ feet high. After how many minutes would the bottom of the tank appear to be on the level of the ground to a person in a train receding from the tank at the rate of 36 miles an hour?

Ex. 150. If C be any fixed point on the line AB produced, the tangents from C to all circles through A and B are equal to one another.

Ex. 151. OAB is an isosceles triangle having OA equal to OB. A line is drawn through O cutting AB in P and the circumference of the circumscribing circle in Q; shew that the rectangle OP, OQ is the same for all directions of the line.

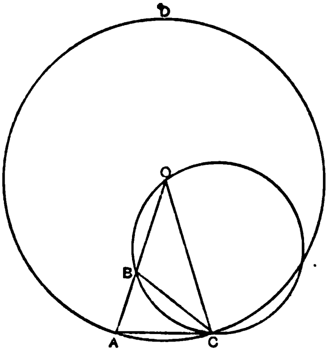
Ex. 152. If two straight lines AB, CD cut one another internally or externally at O so that the rectangle OA, OB is equal to the rectangle OC, OD, shew that a circle may be drawn through A, B, C, D.

Ex. 153. Tangents OA, OB are drawn from an external point to a circle, PQ is any chord through the middle point of AB: shew that the line joining O to the centre bisects the angle POQ.

Ex. 154. OA, OB are tangents from a point O to a circle. C is the centre, D the middle point of AB, PQ is any chord through O: shew that AB bisects the angle PDQ.

PROB. 13. **To inscribe in a circle a regular decagon; and thence to circumscribe a regular decagon about a circle; also to inscribe in, or to circumscribe about, a given circle a regular pentagon, or regular figures of 20, 40, 80 . . . sides.**

Let ACD be the given circle:
it is required to inscribe a regular decagon in ACD.



Find the centre O, and draw any radius OA.
Divide OA at B, so that the rectangle contained by OA and AB may be equal to the square on OB, *II. Prob.* 6.
with centre A and radius equal to OB draw a circle cutting the circumference of ACD at C, join AC:
then shall AC be the side of a regular decagon inscribed in ACD.

Join OC, BC, and draw a circle through the three points O, B, C. *III. Prob.* 6.

Then, because the square on AC is equal to the rectangle contained by AO and AB, *Constr.*
therefore AC touches the circle OBC; *III.* 28, *Cor.* 4.
therefore the angle ACB is equal to the angle BOC in the alternate segment of the circle; *III.* 23.
therefore the whole angle ACO is equal to the sum of the angles BOC and BCO, *Ax. d.*
and therefore to the exterior angle ABC. *I.* 25.

But, because OA is equal to OC;
therefore the angle OAC is equal to the angle OCA, *I.* 7.
and therefore to the angle ABC, *Ax. c.*
therefore BC is equal to AC, *I.* 8.
and therefore to OB; *Ax. c.*
therefore the angle BCO is equal to the angle BOC. *I.* 7.
Hence the angle ACO is equal to the sum of two angles each of which has been proved equal to the angle AOC,
therefore also the angle OAC is equal to twice the angle AOC,
and therefore AOC is a fifth-part of the sum of the angles of the triangle OAC,
and therefore a tenth part of four right angles;
and therefore the minor arc AC is a tenth part of the whole circumference ACD. *III.* 4.

Divide the whole circumference ACD into arcs equal to the minor arc AC, and draw the chords of these arcs; the polygon so formed is the regular inscribed decagon required. *III.* 26.

Also, by drawing tangents at the points at which the circumference is divided, a regular circumscribed decagon is obtained. *III.* 26.

By taking alternate points of division the circumference is divided into five equal parts, and the regular inscribed and circumscribed pentagons may be drawn.

Figures of 20, 40, 80 . . . sides may be obtained by repeatedly bisecting the arcs equal to AC, and drawing chords and tangents at the points of division.

Q.E.F.

Ex. 155. If the two circles intersect again at D, the chord CD is equal to AC. Also OB, BC, CD are three sides of a regular pentagon inscribed in the circle OBC.

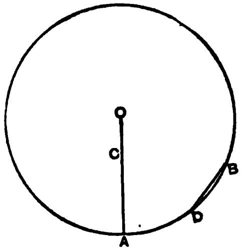
Ex. 156. Describe an isosceles triangle having each of the angles at the base one-third of the vertical angle.

**Ex.* 157. Divide a right angle into five equal parts.

Ex. 158. Shew that each diagonal of a pentagon is parallel to one of the sides.

PROB. 14. **To inscribe in a circle a regular quindecagon, and thence to circumscribe a regular quindecagon about a circle; also to inscribe in, or to circumscribe about, a given circle regular figures of 30, 60, 120 . . . sides.**

Let ADB be the given circle:
it is required to inscribe a regular quindecagon in ADB.



Find the centre O, and draw any radius OA.
Divide OA at C so that the square on AC may be equal to the

rectangle contained by AO and OC; *II. Prob.* 6.
with centre A and radii AO and AC describe circles cutting the circumference of the circle ADB at B and D;
join BD:
then shall BD be the side of a regular quindecagon inscribed in ADB.

For AB is the side of a regular hexagon inscribed in ADB, *III. Prob.* 12.
and AD the side of a regular inscribed decagon; *III. Prob.* 13.
therefore of such parts as the whole circumference contains thirty,
the arc AB contains five, and the arc AD three,
and therefore the arc BD contains two;
therefore the arc BD is a fifteenth part of the whole circumference.

Divide the whole circumference into arcs equal to the arc BD, and draw the chords of these arcs, the polygon so formed is the regular inscribed quindecagon required. *III.* 26.

Also by drawing tangents at the points at which the circumference is divided, a regular circumscribed quindecagon is obtained. *III.* 26.

Figures of 30, 60, 120 . . . sides may be obtained by repeatedly bisecting the arcs equal to BD, and drawing chords, and tangents at the points of division.

Q.E.F.

EXERCISES.

159. Divide a given straight line so that the rectangle contained by its segments shall be equal to a given square not greater than that on half the line.

160. Find a point in the base of a triangle such that the square on the straight line drawn to the vertex may be equal to the rectangle contained by the segments of the line. Distinguish the cases in which there are two solutions, one, or none.

161. If one side of a pentagon be produced, trisect the external angle.

162. If with one of the angular points of a regular pentagon as centre and one of its diagonals as radius a circle be described; a side of the pentagon will be equal to the side of the regular decagon inscribed in the circle.

163. The centre of each of two equal circles lies on the circumference of the other. Shew that the square on the common chord is three times that on the radius.

164. If from a point without a circle there be drawn two straight lines, one of which is perpendicular to a diameter, and the other cuts the circle: the square on the perpendicular is equal to the sum or difference of the rectangle contained by the whole cutting line and the part without the circle and the rectangle contained by the segments of the diameter according as the diameter is divided internally or externally by the perpendicular.

EXERCISES ON BOOK III.

*165. If the perpendicular AD on the side BC of the triangle ABC meets the circumscribed circle of the triangle at P, and if either of the perpendiculars from B or C on the opposite side of the triangle meet AD at O; shew that OD is equal to DP.

*166. The perpendiculars from the vertices of a triangle on the opposite sides meet in a point (*called the* **orthocentre** *of the triangle*).

*167. If O is the orthocentre of the triangle ABC, then any one of the four points O, A, B, C is the orthocentre of the triangle whose vertices are the other three.

*168. The rectangles contained by the segments into which the orthocentre divides each of the perpendiculars from the vertices of a triangle on the opposite sides are equal to one another.

*169. If three straight lines drawn from the vertices of a triangle to the opposite sides meet in a point and are divided at that point into segments containing equal rectangles, shew that the point must be the orthocentre of the triangle.

*170. If O is the orthocentre of the triangle ABC, the angles BOC, COA, AOB are either supplementary or equal to the angles A, B, C respectively.

*171. If any one of four points be the orthocentre of the triangle formed by the other three, shew that the circles through any three of the points are equal to one another.

*172. The distances of the orthocentre of the triangle

ABC from A, B, C are respectively double of the distances of the centre of the circumscribed circle from BC, CA, AB.

*173. AD, BE, CF are drawn perpendicular to the sides BC, CA, AB of a triangle ABC. If the triangle is acute-angled shew that these perpendiculars bisect the angles of the triangle DEF. What is the corresponding theorem when the triangle is obtuse-angled?

*174. If O is the orthocentre of an acute-angled triangle ABC, D, E, F the feet of the perpendiculars from the vertices on the opposite sides, shew that O is the centre of the inscribed, and A, B, C the centres of the escribed circles of the triangle DEF.

*175. The circle through the middle points of the sides of a triangle ABC also passes through the feet of the perpendiculars from the vertices on the opposite sides. Hence if O be the orthocentre the same circle also passes through the middle points of OA, OB, OC. (*The circle is called the* **nine-points circle** *of the triangle.*)

*176. The centre of the nine-points circle of a triangle bisects the straight line joining the orthocentre and the centre of the circumscribed circle.

*177. The diameter of the nine-points circle of a triangle is half that of the circumscribed circle.

178. Find the locus of a point from which equal tangents can be drawn to two given intersecting circles.

179. If the three common chords of three circles which intersect each other, two and two, be produced, they will intersect in a point.

180. If three circles touch two and two, the tangents at the points of contact meet at a point and are equal.

181. The tangents at the points A, B on a circle whose

centre is O intersect in D. Prove that the tangents at the extremities of all chords of the circle which are bisected by AB intersect upon the circle whose diameter is OD.

*182. The feet of the perpendiculars from any point on the circumference of the circumscribed circle of a triangle on the sides of the triangle lie in a straight line.

*183. Describe a circle through two given points and touching a given straight line.

*184. Describe a circle through two given points and touching a given circle.

185. A triangle has two of its angles each double the third. Prove that the circle through the middle points of its sides will intercept portions from the equal sides which are sides of a regular pentagon inscribed in the circle.

186. Construct a square such that two of its sides shall pass through two given points, and the remaining two intersect in another given point.

187. A quadrilateral can be inscribed in and circumscribed about a circle; prove that the straight lines joining the opposite points of contact of the inscribed circle are at right angles to each other.

188. If perpendiculars be drawn from the extremities of the diameter of a circle on any chord or any chord produced, the rectangle contained by the segments between the feet of the perpendiculars and either extremity of the chord is equal to that contained by the perpendiculars.

189. Two points D, E are taken in the base of a triangle ABC so that the tangents drawn from the points B, C to the circle circumscribing ADE are equal: shew that this requires that D and E shall be equidistant from B and C respectively.

190. The four sides of any quadrilateral inscribed in a circle being given, shew that the area does not depend upon the order in which the sides are placed in the circle. Will the order of the sides affect the magnitude of the angles of the figure?

191. The circumference of one circle passes through the centre of another circle. If from any point in the circumference of the former circle two straight lines be drawn to touch the latter circle, prove that the straight line joining the points of contact is bisected by the common chord of the circles.

192. If a regular pentagon ABCDE be inscribed in a circle and P be the middle point of the arc AB, prove that AP and the radius of the circle are together equal to PC.

193. When four circles can be described to touch three given straight lines the square on the distance between the centres of any two together with that on the distance between the centres of the other two is equal to the square on the diameter of the circle through the centres of any three.

194. If P be a point in a diameter of a circle and PT the perpendicular on the tangent at a point Q, then rectangle PT, AB is equal to the rectangle AP, PB together with the square on PQ.

DEFINITIONS OF BOOK III.

DEF. 1. *A* **circle** *is a plane figure contained by one line, which is called the* **circumference,** *and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the* **centre** *of the circle.*

DEF. 2. *A* **radius** *of a circle is a straight line drawn from the centre to the circumference.*

DEF. 3. *A* **diameter** *of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.*

DEF. 4. *An* **arc** *is a part of a circumference. Two arcs which together make the whole circumference are said to be* **conjugate.** *The greater of the two is called the* **major conjugate** *arc and the smaller the* **minor conjugate** *arc.*

DEF. 5. *The conjugate angles formed at the centre of a circle by two radii are said to* **stand** *upon the conjugate arc opposite them intercepted by the radii, the major angle upon the major arc, and the minor angle upon the minor arc.*

DEF. 6. *A* **sector** *is a figure contained by an arc and the radii drawn to its extremities. The* **angle of the sector** *is the angle at the centre which stands upon the arc of the sector.*

DEF. 7. *A* **chord** *of a circle is the straight line joining any two points on the circumference.*

DEF. 8. *A* **secant** *is a straight line of unlimited length which meets the circumference of a circle in two points.*

DEF. 9. *A* **segment** *of a circle is the figure contained by a chord and either of the arcs into which the chord divides the circumference. The segments are called* **major** *or* **minor** *segments according as the arcs that bound them are major or minor arcs.*

DEF. 10. *The angle formed by any two chords drawn from a point on the circumference of a circle is called an* **angle at the circumference,** *and is said to stand upon the arc between its arms.*

DEF. 11. *An angle contained by two straight lines drawn from a point in the arc of a segment to the extremities of the chord is called an* **angle in the segment.**

DEF. 12. *If the angular points of a rectilineal figure be in the circumference of a circle, the figure is said to be* **inscribed** *in the circle, and the circle to be* **circumscribed** *about the figure.*

DEF. 13. *The straight line which, meeting the circumference of a circle in one point, does not meet it again, is said to* **touch,** *or to be a* **tangent** *to, the circle at the point. The point is called the* **point of contact.**

DEF. 13. Aliter. *If a secant of a circle alters its position in uch a manner that the two points of intersection continually approach, and ultimately coincide with one another, the secant in its limiting position is said to* **touch,** *or to be a* **tangent** *to, the circle. The point in which the two points of intersection ultimately coincide is called the* **point of contact.**

DEF. 14. *Two circles are said to* **touch externally** *at a point where they meet, if each lies outside the other; to* **intersect** *if a part of each lies inside, and the remaining part outside the other; and to* **touch internally** *at a point where they meet, if one of them lies inside the other.*

DEF. 15. *The circle that touches the three sides of a triangle is called the* **inscribed** *circle of the triangle.*

DEF. 16. *A circle that touches one side of a triangle and the other two sides produced is called an* **escribed** *circle of the triangle.*

DEF. 17. *If all the sides of a rectilineal figure touch a circle lying within the figure, the circle is said to be* **inscribed** *in the figure, and the figure is said to be* **circumscribed** *about the circle.*

BOOK IV.

FUNDAMENTAL PROPOSITIONS OF PROPORTION.

INTRODUCTORY REMARKS.

IN this book the subjects of the propositions, in the first Sections of both Parts I. and II., on Ratio and Proportion, are magnitudes in general, and not merely geometrical magnitudes—such as lines, angles, areas, etc. Many of the propositions are familiar, under somewhat different forms, to those who have studied Arithmetic and Elementary Algebra; but, as the starting-point of these latter sciences is the notion of *number* and not of *magnitude in general*, it is important that the more general treatment of Ratio and Proportion, either in the exact and complete form of Part II., or in the equally exact but restricted, and thereby simplified, form of Part I., should be mastered before proceeding to their application to Geometry.

A *Magnitude* is anything of which a greater or a less can be predicated.

Thus, number, length, area, value, probability, time, speed, weight, etc., and so also hunger, love, courage, talent, wisdom, etc., are magnitudes, each different in kind from each of the others, and no one comparable with another of a different kind as to quantity or intensity. On the other hand, position, direction, relation, etc., are not magnitudes, inasmuch as it is unmeaning to speak of a greater or less position, direction, etc., unless in some conventional or specially defined sense of the word.

In order that any magnitude may be made the subject of exact mathematical treatment, it must be possible to define with exactness the *equality* of two amounts of such magnitude.

In Geometry the test of equality is given in the Axiom (I, Ax. 1), "Magnitudes that can be made to coincide are equal." Corresponding tests of equality form the foundation of the sciences which deal with value, probability, time, speed, weight, etc. It is the impossibility of giving exact tests of the equality of different amounts of hunger, love, courage, talent, etc., or any moral qualities, which prevents their forming the subject of exact measurement, and so of mathematical treatment.

Number and magnitudes that are completely expressible by number differ from other magnitudes in that they are essentially *discrete* or discontinuous. Fundamentally, number is *counting*, and to count is to add unit after unit, and so to increase by jumps or finite additions, while magnitudes in general are *continuous*, and increase insensibly from one value to another through every intermediate value.

Thus the population of a country is a magnitude which increases or diminishes by the addition or removal of individual human beings, while the height or weight of a man is a magnitude which increases or diminishes by *continuous* change or *growth*.

It is impossible, therefore, to represent continuous magnitude exactly by number, though number can obviously be represented by continuous magnitude.

Magnitudes which are *exactly* expressible by numbers of the same common unit, are said to be *commensurable:* while, if there is no common unit by which they can be exactly expressed, they are said to be *incommensurable.*

To prove the existence of incommensurable quantities, let us consider the diagonal and side of a square.

If the diagonal and side of a square can be measured or expressed in numbers of a common unit, suppose that the one contains the greatest unit by which they can be so measured m times and the other n times. Then m and n cannot *both* be even numbers, for in that case the two lines could be measured by double the supposed greatest unit. But since the square on the diagonal contains m^2 square units, and that on the side n^2 square units, and we know (II. 9) that the square on the diagonal is double the square on the side, therefore $m^2 = 2n^2$. Hence m^2 is an even number, and, therefore, it is the square of an even number, so that m is even. Let $m = 2p$; then, since $2n^2 = m^2 = 4p^2$, therefore $n^2 = 2p^2$, whence also n is an even number. But we have seen that m and n cannot *both* be even, therefore, there can be no common unit by which the diagonal and side of a square can be expressed, so that they are *incommensurable.*

In fact, it should be observed that with regard to different quantities of the same continuous magnitude, incommensurability is the rule, and commensurability the exception.

For practical purposes, by taking a sufficiently small unit, every continuous magnitude may be expressed by means of a number, but it must be clearly understood that such expression is not exact, but approximate only.

Thus it is true that of the parts of which the side of a square contains 100, the diagonal contains more than 141, and less than

142, and if $\frac{1}{100}$ of the side be regarded as sufficiently small to be neglected, the diagonal is measured approximately by 141. If greater accuracy is required, but $\frac{1}{1000}$ may be neglected, it is measured approximately by 1,414 parts, of which the side contains 1,000: and so on to any degree of accuracy that may be required.

A theory of Proportion, incomplete but sufficient for practical applications, is given in Part I. of this book, which treats of Commensurable Magnitudes only; but the student who aims at mastering a complete theory for magnitudes in general, without regard to the distinction of commensurable and incommensurable, must study Part II., which, though different in the form of expression, is essentially the theory contained in the Fifth Book of Euclid's Elements.

PART I.

FUNDAMENTAL PROPOSITIONS OF PROPORTION FOR COMMENSURABLE MAGNITUDES ONLY.

SECTION I.

OF RATIO AND PROPORTION.

[*Notation.*

In what follows, large Roman letters, A, B, etc., are used to denote magnitudes, and where the pairs of magnitudes compared are both of the same kind, they are denoted by letters taken from the early part of the alphabet, as A, B compared with C, D; but where they are, or may be, of different kinds, from different parts of the alphabet, as A, B compared with P, Q or X, Y. Small italic letters, m, n, etc., denote whole numbers. By $m.\mathrm{A}$ or $m\mathrm{A}$ is denoted the mth multiple of A, and it may be read as m times A. The product of the numbers m and n is denoted by mn, and it is assumed that $mn = nm$. The combination $m.n\mathrm{A}$ denotes the mth multiple of the nth multiple of A, and may be read as m times $n\mathrm{A}$, and $mn\mathrm{A}$ or $mn.\mathrm{A}$ as mn times A. By $(m+n)\,\mathrm{A}$ is denoted $m+n$ times A.]

DEF. 1. *One magnitude is said to be a* **multiple** *of another magnitude when the former contains the latter an exact number of times. According as the number of times is* 1, 2, 3...m, *so is the multiple said to be the* 1*st*, 2*nd*, 3*rd*,...mth.

DEF. 2. *One magnitude is said to be* **a measure** or **part** *of another magnitude when the former is contained an exact number of times in the latter.*

The following properties of multiples will be assumed:—

1. If A=B, then mA=mB.
2. If mA=mB, then A=B.
3. $m\mathrm{A}+m\mathrm{B}+\ldots=m\,(\mathrm{A}+\mathrm{B}+\ldots)$.
4. $m\mathrm{A}-m\mathrm{B}=m\,(\mathrm{A}-\mathrm{B})$, (A being greater than B).
5. $m\mathrm{A}+n\mathrm{A}=(m+n)\mathrm{A}$.
6. $m\mathrm{A}-n\mathrm{A}=(m-n)\mathrm{A}$, ($m$ being greater than n).
7. $m.n\mathrm{A}=mn.\mathrm{A}=nm.\mathrm{A}=n.m\mathrm{A}$.

THEOR. 1. **If two magnitudes have a common multiple they have also a common measure.**
Conversely, if two magnitudes have a common measure they have also a common multiple.

Let A and B have a common multiple C,
i.e., let C=mA or nB.
Suppose C to contain mn parts each equal to D.
Because C=mA, and C=mnD,
therefore mA=mnD
=$m.n$D.
Therefore A=nD. *Props. of Mults.* 2.
In like manner B=mD.
Hence A and B have a common measure D.

Conversely, let A and B have a common measure D,
i.e., let A=mD, and B=nD.

Take C the mnth multiple of D.
Then $C=mnD=n.mD=nA$.
In like manner $C=mnD=m.nD=mB$.

Hence A and B have a common multiple C.

Q.E.D.

DEF. 3. *If two or more magnitudes have a common multiple or measure they are said to be* **commensurable.**

DEF. 4. *The* **ratio** *of one magnitude to another of the same kind is a certain relation of the former to the latter in respect of quantity, the comparison being made by considering what multiples of the two magnitudes are equal to one another. The ratio of* A *to* B *is denoted thus,* A : B, *and* A *is called the* **antecedent,** B *the* **consequent.**

Thus the ratio of a half-crown to a florin is the relation expressed by stating that 4 half-crowns = 5 florins.

DEF. 5. *The ratio $A : B$ is said to be equal to the ratio $P : Q$ when like multiples of A and P (mA and mP) are equal respectively to like multiples of B and Q (nB and nQ), and the four magnitudes are said to be* **proportionals,** *or to form a* **proportion.**

The equality of the ratios is denoted by the symbol : : ; and the proportion thus, $A : B :: P : Q$, which is read, "A is to B as P is to Q."
A and Q are called the **extremes,** *B and P the* **means,** *and Q is said to be the* **fourth proportional** *to A, B, and P.*

The antecedents A, P are said to be **homologous** *to one another, and so also are the consequents B, Q.*

Thus because 4 half-crowns = 5 florins
and 4 shillings = 5 francs

and 4 English ells = 5 yards
and 4 fives = 5 fours
therefore a half-crown : a florin : : a shilling : a franc
: : an English ell : a yard
: : 5 : 4.

It follows immediately from Def. 5, that
if A : B : : P : Q, then B : A : : Q : P. (this inference is usually referred to as *invertendo* or *inversely*);
Also that if A : B : : P : Q and X : Y : : P : Q, then A : B : : X : Y.

DEF. 6. *If A, B, C are three magnitudes of the same kind such that A : B : : B : C, B is said to be the* **mean proportional** *between A and C, and C* the **third proportional** to *A and B.*

THEOR. 2. **Equal magnitudes have the same ratio to the same magnitude.**
Conversely, magnitudes that have the same ratio to the same magnitude are equal.

Let A, B, C be three magnitudes of the same kind, and let A = B.
Suppose mA = nC.
Then because A = B, therefore mA = mB;
but mA = nC, therefore mB = nC.
Hence A : C : : B : C. *Def.* 5.

Conversely, let A : C : : B : C.
Then if mA = nC, mB = nC *Def.* 5.
therefore mA = mB,
and therefore A = B. Q.E.D.

THEOR. 3. **The ratio of two magnitudes is equal to that of their doubles.**

Let A, B be two magnitudes having a ratio such as that $mA = nB$,
then $2.mA = 2.nB$,
but $2.mA = m.2A$, and $2.nB = n.2B$,
therefore $m.2A = n.2B$.
Hence $A : B :: 2A : 2B$. *Def.* 5.
Q.E.D.

COR. **The ratio of two magnitudes is equal to that of their halves.**

For $A : B :: 2A : 2B$, and A and B are the halves of 2A and 2B respectively.

THEOR. 4. **If $A : B :: C : D$, all the four magnitudes being of the same kind, then $A : C :: B : D$.**

(This inference is usually referred to as *alternando* or *alternately.*)

Suppose A, B to be such that $mA = nB$, then $m C = nD$. *Def.* 5.
Suppose also that $pA = qC$.
Because $mA = nB$,
therefore $p.mA = p.nB$, *Props. of Mults.* 1.
therefore $m.pA = n.pB$. *Props. of Mults.* 7.
In like manner, $m.qC = n.qD$.
But $m.pA = m.qC$, since $pA = qC$,
therefore $n.pB = n.qD$,
and therefore $pB = qD$. *Props. of Mults.* 2.

Hence, because $pA=qC$, and $pB=qD$,
therefore $A : C :: B : D$; *Def.* 5.
or, if four magnitudes of the same kind be proportionals, the first has to the third the same ratio as the second to the fourth.

Q.E.D.

THEOR. 5. **If** $A : B :: P : Q$,
then, $A+B : B :: P+Q : Q$,
and $A-B : B :: P-Q : Q$.

(These inferences are referred to as *componendo* and *dividendo* respectively.)

Suppose A, B to be such that $mA=nB$, then $mP=nQ$. *Def.* 5.
Because $mA=nB$,
therefore $mA+mB=nB+mB$,
therefore $m(A+B)=(n+m)B$. *Props. of Mults.* 3 and 5.
In like manner $m(P+Q)=(n+m)Q$.
Hence $A+B : B :: P+Q : Q$. *Def.* 5.

By like reasoning (A being greater than B)
$A-B : B :: P-Q : Q$.

Q.E.D.

THEOR. 6. **If** $A : B :: C : D :: E : F$,
then $A : B :: A+C+E : B+D+F$.

(This inference is referred to as *addendo.*)

Suppose A, B to be such that $mA = nB$, then $mC = nD$, and $mE = nF$. *Def.* 5.
Then $mA+mC+mE=nB+nD+nF$,

therefore $m(A+C+E) = n(B+D+F)$. *Props. of Mults.* 3.
Hence $A : B :: A+C+E : B+D+F$. *Def.* 5.
or, if any number of magnitudes of the same kind be proportionals, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents.

Q.E.D.

COR. **If** $A : B :: C : D$ (A **being greater than** C), **then** $A : B :: A - C : B - D$.

THEOR. 7. **If** $A : B :: P : Q$,
and $B : C :: Q : R$,
then $A : C :: P : R$.
(This inference is referred to as *ex æquali.*)

Suppose A, B to be such that $mA = nB$, then $mP = nQ$; *Def.* 5.
suppose also B, C to be such that $hB = kC$, then $hQ = kR$. *Def.* 5.
Because $mA = nB$,
therefore $h.mA = h.nB$, *Props. of Mults.* 1.
therefore $mh.A = nh.B$.
So also, $nh.B = nk.C$,
therefore $mh.A = nk.C$.
In like manner $mh.P = nk.R$.
Hence $A : C :: P : R$; *Def.* 5.
or, if there be two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other: then the first is to the third of the first set as the first to the third of the other.

Q.E.D.

THEOR. 8. **If** A : C : : P : R,
and B : C : : Q : R,
then A+B : C : : P+Q : R.

Because B : C : : Q : R,	*Hyp.*
therefore C : B : : R : Q,	*Invertendo.*
and A : C : : P : R,	*Hyp.*
therefore A : B : : P : Q,	*Ex æquali.*
therefore A+B : B : : P+Q : Q,	*Componendo.*
and B : C : : Q : R	
therefore A+B : C : : P+Q : R.	*Ex æquali.*
	Q. E. D.

DEF. 7. *If there are three magnitudes A, B, C of the same kind, then A is said to have to C the ratio* **compounded** *of the ratios A : B, B : C.*

DEF. 8. *If there are two ratios A : B, P : Q, and C be taken such that B : C : : P : Q, then A is said to have to C the ratio* **compounded** *of the ratios A : B, P : Q.*

DEF. 9. *A ratio compounded of two equal ratios is called the* **duplicate** *of either of these ratios.*

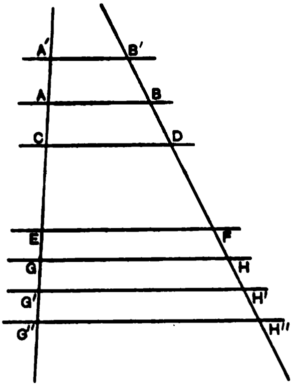
From Def. 7 it follows that the ratio compounded of A : B and B : A is the ratio A : A;
or, the ratio compounded of a given ratio and its reciprocal is the ratio of equality.

SECTION II.

FUNDAMENTAL GEOMETRICAL PROPOSITIONS.

THEOR. 9. **If there are two pairs of straight lines all of which are parallel, then the intercepts made by each pair on a straight line that cuts them are one to another in the same ratio as the corresponding intercepts on any other straight line that cuts them.**

Let AB, CD and EF, GH be two pairs of straight lines all of which are parallel, and let AG, BH be two straight lines that cut them :



then shall AC be to EG as BD to FH,

Suppose 2AC=3EG.
On AG take AA′ equal to AC, so that CA′=2AC, and GG′, G′G″ each equal to EG, so that EG″=3EG.
Draw A′B′, G′H′, G″H″ parallel to AB to meet BH in B′, H′, H″ respectively.

Then, because AC=AA′,
therefore BD=BB′, *I.* 34.
and therefore DB′=2BD.
In like manner FH″=3FH.

But, by supposition, CA′=EG″,
therefore DB′=FH″ *I.* 34.
or, 2BD=3FH.

If m.AC=n.EG, we can prove by like reasoning that m.BD =n.FH.
Hence AC : EG : : BD : FH. *Def.* 5.
Q.E.D.

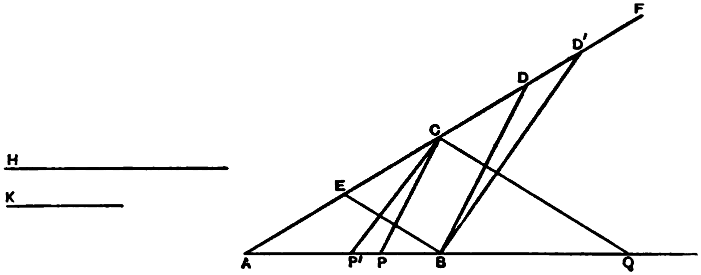
COR. 1. **If there are three parallel straight lines, the intercepts made by them on any straight line that cuts them are to one another in the same ratio as the corresponding intercepts on any other straight line that cuts them.**

COR. 2. **If the sides of a triangle are cut by a straight line parallel to the base, the segments of one side are to one another in the same ratio as the segments of the other side.**

THEOR. 10. **A given finite straight line can be divided internally into segments having any given ratio, and also**

externally into segments having any given ratio except the ratio of equality; and in each case there is only one such point of division.

Let AB be the given straight line,
H, K two straight lines in the given ratio:



then can AB be divided internally and externally in the ratio H : K; and in each case in one point only.

From A draw the straight line AF making an angle with AB. From AF cut off AC equal to H, and CD, CE each equal to K; join BD, BE, and from C draw CP parallel to BD, and CQ to BE, meeting AB internally and externally at P and Q.

Because PC is parallel to BD,
therefore AP : PB : : AC : CD *IV.* 9, *Cor.* 2.
therefore AP : PB : : H : K.

Again, because QC is parallel to BE,
therefore AQ : QB : : AC : CE,
therefore AQ : QB : : H : K.

Hence AB is divided internally at P and externally at Q in the given ratio H : K.

Also AB cannot be divided internally in the ratio H : K at any point other than P.
For, if possible, let P′ be a point such that AP′ : P′B : : H : K.
Join CP′, and from B draw BD′ parallel to CP′ and meeting AF at D′.
Then, because CP′ is parallel to D′B,
therefore AC : CD′ : : AP′ : P′B *IV.* 9, *Cor.* 2.
but AP′ : P′B : : H : K : : AC : CD
therefore AC : CD′ : : AC : CD
therefore CD′=CD *IV.* 2.
which is impossible.
Hence P is the only point which divides AB internally in the given ratio.

A like demonstration will shew that Q is the only point which divides AB externally in the given ratio.

Q.E.D.

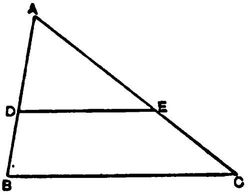
**Ex.* 1. Trace the changes in position of P and Q, as K increases from being much smaller than H, till it is equal to H, and then till it is much larger than H.

DEF. 10. *When a line AB is divided internally at P and externally at Q in the same ratio, it is said to be* **divided harmonically** *by the points P and Q; and the points A, P, B, Q are said to form an* **harmonic range.**

* *Ex.* 2. If A, P, B, Q form an harmonic range, and M is the middle point of AB, prove that MA or MB is the mean proportional between MP and MQ.

THEOR. 11. **A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle.**

Let the straight line DE divide the sides AB, AC of the triangle ABC, so that AD : DB : : AE : EC,



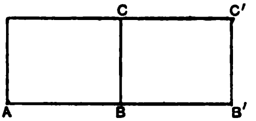
then shall DE be parallel to BC.

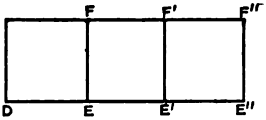
For the parallel to BC through D divides AC in the ratio AD : DB *IV.* 9, *Cor.* 2.
and therefore must pass through E, the only point in which AC is divided in this ratio; *IV.* 10.
therefore DE is parallel to BC.

Q.E.D.

THEOR. 12. **Rectangles of equal altitude are to one another in the same ratio as their bases.**

Let AC, DF be two rectangles of equal altitude on the bases AB, DE :





then shall the rectangle AC be to the rectangle DF as the base AB to the base DE.

Suppose $2AB = 3DE$.
On AB produced take BB′ equal to AB, so that $AB' = 2AB$;
and on DE produced take EE′, E′E″ each equal to DE,
so that $DE'' = 3DE$.
Complete the rectangles BC′, EF′, E′F″.

Then because $AB = BB'$,
therefore the rectangle AC = the rectangle BC′, *I.* 30, *Cor.*
and therefore the rectangle AC′ = twice the rectangle AC.
In like manner the rectangle DF″ = three times the rectangle DF.

But, by supposition, $AB' = DE''$,
therefore the rectangle AC′ = the rectangle DF″, *I.* 30, *Cor.*
or, twice the rectangle AC = three times the rectangle DF.

If m times $AB = n$ times DE, we can prove by like reasoning that m times the rectangle $AC = n$ times the rectangle DF.

Hence the rectangle AC is to the rectangle DF as the base AB to the base DE.

Q.E.D.

Cor. **Parallelograms or triangles of the same altitude are to one another as their bases.**

For a parallelogram is equal to a rectangle whose base and altitude are equal to those of the parallelogram. *II.* 1, *Cor.* 1.

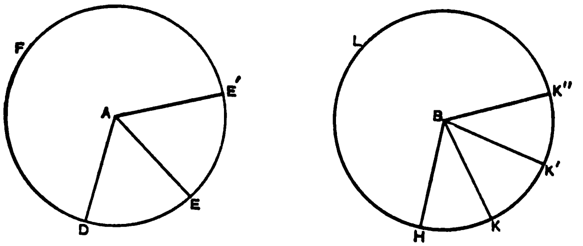
Also a triangle is equal to half a rectangle whose base and altitude are equal to those of the triangle (*II.* 2), and the ratio of two magnitudes is equal to that of their doubles. *IV.* 3.

**Ex.* 3. Parallelograms or triangles on equal bases are to one another as their altitudes.

THEOR. 13. **In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand.**

[NOTE.—In Book III. it was only necessary to consider arcs less than the whole circumference, and angles less than four right angles; but Theors. 4 and 5, Book III., are equally true for arcs greater than one or any number of circumferences, and the corresponding angles greater than four right angles.]

In the equal circles DEF, HKL, of which A and B are the respective centres, let the angles DAE, HBK stand on the arcs DE, HK:



then shall the angle DAE be to the angle HBK as the arc DE is to the arc HK.

Suppose twice the arc DE=three times the arc HK. On the circumference DEF take the arc EE′ equal to the arc DE, so that the arc DE′=twice the arc DE; and on the circumference HKL take the arcs KK′, K′K″ each equal to the arc HK, so that the arc HK″=three times the arc HK.
Join AE′, BK′, BK″.

Then because the arc DE=the arc EE′,
therefore the angle DAE=the angle EAE′, *III.* 5.
and therefore the angle DAE′=twice the angle DAE.
In like manner the angle HBK″=three times the angle HBK.

But, by supposition, the arc DE′=the arc HK″,
therefore the angle DAE′=the angle HBK″, *III.* 5.
or, twice the angle DAE=three times the angle HBK.

If m times the arc DE=n times the arc HK, we can prove by like reasoning that m times the angle DAE=n times the angle HBK.

Hence the angle DAE is to the angle HBK as the arc DE is to the arc HK.

Q.E.D.

DEFINITIONS OF BOOK IV.

PART I.

DEF. 1. *One magnitude is said to be a* **multiple** *of another magnitude when the former contains the latter an exact number of times. According as the number of times is* 1, 2, 3...*m*, *so is the multiple said to be the* 1*st*, 2*nd*, 3*rd*,...*m*th.

DEF. 2. *One magnitude is said to be a* **measure** *or* **part** *of another magnitude when the former is contained an exact number of times in the latter.*

DEF. 3. *If two or more magnitudes have a common multiple or measure they are said to be* **commensurable.**

DEF. 4. *The* **ratio** *of one magnitude to another of the same kind is a certain relation of the former to the latter in respect of quantity, the comparison being made by considering what multiples of the two magnitudes are equal to one another. The ratio of* A *to* B *is denoted thus,* A : B, *and* A *is called the* **antecedent,** B *the* **consequent.**

Thus the ratio of a half-crown to a florin is the relation expressed by stating that 4 half-crowns = 5 florins.

DEF. 5. *The ratio* $A : B$ *is said to be equal to the ratio* $P : Q$ *when like multiples of* A *and* P (mA *and* mP) *are equal respectively to like multiples of* B *and* Q (nB *and* nQ), *and the four magnitudes are said to be* **proportionals,** *or to form a* **proportion.**

The equality of the ratios is denoted by the symbol : : ; *and the pro-*

portion thus, A : B : : P : Q, which is read "A is to B as P is to Q." A and Q are called the **extremes,** *B and P the* **means,** *and Q is said to be the* **fourth proportional** *to A, B, and P.*

The antecedents A, P, are said to be **homologous** *to one another, and so also are the consequents.*

DEF. 6. *If A, B, C are three magnitudes of the same kind such that A : B : : B : C, B is said to be the* **mean proportional** *between A and C, and C* the **third proportional** *to A and B.*

DEF. 7. *If there are three magnitudes A, B, C, of the same kind, then A is said to have to C the ratio* **compounded** *of the ratios A : B, B : C.*

DEF. 8. *If there are two ratios A : B, P : Q, and C be taken such that B : C : : P : Q, then A is said to have to C the ratio* **compounded** *of the ratios A : B, P : Q.*

DEF. 9. *A ratio compounded of two equal ratios is called the* **duplicate** *of either of these ratios.*

DEF. 10. *When a line AB is divided internally at P and externally at Q in the same ratio, it is said to be* **divided harmonically** *by the points P and Q; and the points A, P, B, Q are said to form an* **harmonic range.**

PART II.

FUNDAMENTAL PROPOSITIONS OF PROPORTION FOR MAGNITUDES, WITHOUT RESPECT TO COMMENSURABILITY.

SECTION I.

OF RATIO AND PROPORTION.

[*Notation.*

In what follows, large Roman letters A, B, etc., are used to denote magnitudes, and where the pairs of magnitudes compared are both of the same kind, they are denoted by letters taken from the early part of the alphabet, as A, B compared with C, D; but where they are or may be of different kinds, from different parts of the alphabet, as A, B compared with P, Q or X, Y. Small italic letters m, n, etc., denote whole numbers. By $m.$A or mA is denoted the mth multiple of A, and it may be read as m times A. The product of the numbers m and n is denoted by mn, and it is assumed that $mn=nm$. The combination $m.n$A denotes the mth multiple of the nth multiple of A, and may be read as m times nA, and mnA or $mn.$A as mn times A. The symbol $>$ denotes *greater than*, and $<$ *less than*.]

DEF. 1.—*One magnitude is said to be a* **multiple** *of another magnitude, when the former contains the latter an exact number of times. According as the number of times is* 1, 2, 3 . . . m, *so is the multiple said to be the* 1*st*, 2*nd*, 3*rd* . . . mth.

DEF. 2.—*One magnitude is said to be a* **measure** *or* **part** *of another magnitude, when the former is contained an exact number of times in the latter.*

The following properties of multiples will be assumed :—

1. As $A > =$ or $< B$, so is $m\,A > =$ or $< mB$.

The converse necessarily follows (Introd., p. 14), so that

2. As $mA > =$ or $< mB$, so is $A > =$ or $< B$.

3. $mA + mB + \ldots = m(A + B + \ldots)$.
4. $mA - mB = m(A - B)$, (A being greater than B).
5. $mA + nA + \ldots = (m + n + \ldots)\,A$.
6. $mA - nA = (m - n)\,A$, (m being greater than n).
7. $m.nA = mn.A = nm.A = n.mA$.

DEF. 3.—*The* **ratio** *of one magnitude to another of the same kind is the relation of the former to the latter in respect of* **quantuplicity**.

The *quantuplicity* of A with respect to B is estimated by examining how the multiples of A are distributed among the multiples of B, when both are arranged in ascending order of magnitude, and the series of multiples continued without limit.

Thus, if A denotes the diagonal, and B the side of a square, the scale of interdistribution of their multiples begins thus :

A 2A 3A 4A 5A 6A 7A 8A 9A 10A
B 2B 3B 4B 5B 6B 7B 8B 9B 10B 11B 12B 13B 14B

100A
15B . . . . 141B 142B, etc., etc.,

each multiple of A being placed over the interval between the multiples of B, between which it lies.

This interdistribution of multiples is definite for two given magnitudes A and B, and is different from that for A and C, if C differ from B by any magnitude however small.

For if B and C differ by a magnitude D, m may be found so great that $mD > A$, however small D be, and then mB and mC would differ by more than A, and, therefore, could not lie between the same two multiples of A, so that the interdistribution of the multiples of A and B would for high multiples (if not for lower ones) differ from that for A and C.

The ratio of A to B is denoted thus A : B, and A is called the *antecedent*, B the *consequent*, of the ratio.

DEF. 4. *The ratio of one magnitude to another is equal to that of a third magnitude to a fourth (whether of the same or of a different kind from the first and second), when any equimultiples whatever of the antecedents of the ratios being taken and likewise any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as that of the other antecedent is greater than, equal to, or less than that of its consequent.*

Or in other words :
The ratio of A to B is equal to that of P to Q, when mA is greater than, equal to, or less than nB, according as mP is greater than, equal to, or less than nQ, whatever whole numbers m and n may be.
It is an immediate consequence that :
The ratio of A to B is equal to that of P to Q, when, m being any number whatever, and n another number determined so that either mA is between nB and $(n + 1)$ B or equal to nB, according as mA is between nB and $(n + 1)$ B or is equal to nB, so is mP between nQ and $(n + 1)$ Q or equal to nQ.
The definition may also be expressed thus :
The ratio of A to B is equal to that of P to Q when the multiples of A are distributed among those of B in the same manner as the multiples of P are among those of Q.

DEF. 5. *The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when equimultiples of the* *antecedents and equimultiples of the consequents can be found such that, while the multiple of the antecedent of the first is greater than, or equal to, that of its consequent, the multiple of the antecedent of the other is not greater, or is less, than that of its consequent.*

Or in other words:
The ratio of A to B is greater than that of P to Q, when whole numbers m and n can be found, such that, while mA is greater than nB, mP is not greater than nQ, or while $m\text{A} = n\text{B}$, mP is less than nQ.

DEF. 6. *When the ratio of A to B is equal to that of P to Q the four magnitudes are said to be* **proportionals** *or to form a* **proportion.** *The proportion is denoted thus:*

$$A : B :: P : Q,$$

which is read, "A is to B as P is to Q." A and Q are called the **extremes,** *B and P the* **means,** *and Q is said to be the* **fourth proportional** *to A, B, and P. The antecedents A, P are said to be* **homologous,** *and so are the consequents, B, Q.*

DEF. 7. *Three magnitudes (A, B, C) of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third; that is, when*

$$A : B :: B : C.$$

In this case C is said to be the **third proportional** *to A and B, and B the* **mean proportional** *between A and C.*

DEF. 8. *The ratio of any magnitude to an equal magnitude is said to be a* **ratio of equality.** *If A be greater than B, the ratio A : B is said to be a* **ratio of greater inequality,** *and the ratio B : A a* **ratio of less inequality.** *Also the ratios A : B and B : A are said to be* **reciprocal** *to one another.*

THEOR. 1. **Ratios that are equal to the same ratio are equal to one another.**

Let $A : B :: P : Q$ and also $X : Y :: P : Q$,
then $A : B :: X : Y$.
Because $A : B :: P : Q$;
if m is any number whatever,
and mP is between nQ and $(n+1)\,Q$, or is equal to nQ;
therefore mA is between nB and $(n+1)\,B$ or is equal to nB.
Def. 4.
And because $X : Y :: P : Q$,
therefore mX is between nY and $(n+1)\,Y$ or is equal to nY.
Def. 4.
Hence mA is between nB and $(n+1)\,B$ or is equal to nB, according as mX is between nY and $(n+1)\,Y$ or is equal to nY, whatever number m represents,
therefore $A : B :: X : Y$. *Def.* 4.
Q.E.D.

THEOR. 2. **If two ratios are equal, as the antecedent of the first is greater than, equal to, or less than its consequent, so is the antecedent of the other greater than, equal to, or less than its consequent.**

Let $A : B :: P : Q$,
then according as A is $> =$ or $<$ B,
so is $P > =$ or $< Q$.
Since, whatever numbers m and n represent,
mA is $> =$ or $< nB$,
according as mP is $> =$ or $< nQ$, *Def.* 4.
if m and n be taken each $= 1$,

$$A \text{ is } > = \text{ or } < B,$$

according as $P \text{ is } > = \text{ or } < Q.$

Q.E.D.

THEOR. 3. **If two ratios are equal, their reciprocal ratios are equal.**

Let $A : B :: P : Q,$

then $B : A :: Q : P.$

(This inference is usually referred to as *invertendo* or *inversely*.)

Since $A : B :: P : Q,$

the multiples of A lie among those of B in the same intervals as those of P lie among those of Q, *Def.* 4.

therefore the multiples of B lie among those of A in the same intervals as those Q lie among those of P,

and therefore $B : A :: Q : P.$ *Def.* 4.

Q.E.D.

Ex. 1.—Prove from Def. 5 that,

if $A : B > P : Q,$

then $B : A < Q : P.$

THEOR. 4. **If the ratios of each of two magnitudes to a third magnitude be taken, the first ratio will be greater than, equal to, or less than the other as the first magnitude is greater than, equal to, or less than the other: and if the ratios of one magnitude to each of two others be taken, the first ratio will be greater than, equal to, or less than the other as the first of the two magnitudes is less than, equal to, or greater than the other.**

Let A, B, C be three magnitudes of the same kind,
then $A : C$ is $>\,=$ or $< B : C$,
according as A is $>\,=$ or $< B$;
and $C : A$ is $>\,=$ or $< C : B$,
according as A is $<\,=$ or $> B$.

If $A = B$, the multiples of A and B being identical, lie among the multiples of C in the same intervals,
therefore $A : C :: B : C$
and $C : A :: B : A$. *Def.* 4.

If $A > B$, m can be found such that mB is less than mA by a magnitude greater than C; hence, if mA is between nC and $(n+1)\,C$, or is equal to nC, mB is less than nC,

therefore $A : C > B : C$; *Def.* 5.
also since $nC > mB$, while nC is not $> mA$,
therefore $C : B > C : A$, *Def.* 5.
or $C : A < C : B$.
If $A < B$, then $B > A$,
and therefore $B : C > A : C$, and $C : B < C : A$;
or, $A : C < B : C$, and $C : A > C : B$.

Q.E.D.

COR. **Conversely** A **is** $>\,=$ **or** $< B$,
according as $A : C$ **is** $>\,=$ **or** $< B : C$;
or as $C : A$ **is** $<\,=$ **or** $> C : B$.

For the rule of conversion holds. *Introd.* § 9.

Ex. 2. To three given magnitudes (A, B, P) there is only one fourth proportional: and to two given magnitudes (A, B) there is only one third proportional and only one mean proportional.

THEOR. 5. **The ratio of equimultiples of two magnitudes is equal to that of the magnitudes themselves.**

Let A, B be two magnitudes,
then $mA : mB :: A : B$.

If p, q be any two numbers,
according as $pA > =$ or $< qB$;
so is $m.pA > =$ or $< m.qB$; *Props. of Mults.* 1.
but $m.pA = p.mA$, and $m.qB = q.mB$, *Props. of Mults.* 7.
therefore $p.mA > =$ or $< q.mB$,
as $pA > =$ or $< qB$;
whence $mA : mB :: A : B$. *Def.* 4.
Q.E.D.

THEOR. 6. **If two magnitudes (A, B) have the same ratio as two whole numbers (m, n), then $nA = mB$: and conversely if $nA = mB$, A has to B the same ratio as m to n.**

Because $A : B :: m : n$, *Hyp.*
and nA and $n.m$ are equimultiples of A and m,
and mB and $m.n$ are equimultiples of B and n;
therefore since $n.m = m.n$,
$nA = mB$. *Def.* 4.

Conversely, if $nA = mB$,
according as $pA > =$ or $< qB$,
so is $n.pA > =$ or $< n.qB$; *Props. of Mults.* 1.

or, $p.nA >= \text{or} < q.nB$;
or, $p.mB >= \text{or} < q.nB$;
or, $p.m >= \text{or} < q.n$;
therefore $A : B :: m : n$. *Def.* 4.

Q.E.D.

COR. **If $A : B :: P : Q$ and $nA=mB$, then $nP=mQ$; whence if A be a multiple, part, or multiple of a part of B, P is the same multiple, part, or multiple of a part of Q.**

THEOR. 7. **If four magnitudes of the same kind are proportionals, the first is greater than, equal to, or less than the third, according as the second is greater than, equal to, or less than the fourth.**

Let A, B, C, D be four magnitudes of the same kind such that $A : B :: C : D$,
then A is $>= \text{or} < C$,
according as B is $>= \text{or} < D$.

If $B=D$, then $A : B :: A : D$, *IV.* 4.
but $A : B :: C : D$; *Hyp.*
therefore $A : D :: C : D$, *IV.* 1.
and therefore $A=C$. *IV.* 4, *Cor.*

If $B > D$, then $A : B < A : D$, *IV.* 4.
but $A : B :: C : D$, *Hyp.*
therefore $C : D < A : D$ or $A : D > C : D$;
and therefore $A > C$. *IV.* 4, *Cor.*

In like manner if $B < D$, it may be proved that $A < C$.

Q.E.D.

THEOR. 8. **If four magnitudes of the same kind are proportionals, the first has to the third the same ratio as the second to the fourth.**

Let A, B, C, D be four magnitudes of the same kind such that
$$A : B :: C : D,$$
then
$$A : C :: B : D.$$
(This inference is usually referred to as *alternando* or *alternately*.)

Because $mA : mB :: A : B,$ *IV.* 5.
and $nC : nD :: C : D;$ *IV.* 5.
therefore $mA : mB :: nC : nD;$ *IV.* 1.
Hence as $mA > = \text{ or } < nC,$
so is $mB > = \text{ or } < nD;$ *IV.* 7.
therefore, since m and n are any numbers whatever,
$$A : C :: B : D.$$

Q.E.D.

Ex. 3. Prove that, if $A : B :: C : D,$
$$mA : nB :: mC : nD.$$

THEOR. 9. **If any number of magnitudes of the same kind are proportionals, as one of the antecedents is to its consequent, so is the sum of the antecedents to the sum of the consequents.**

Let A,C,E . . . and B,D,F . . . be magnitudes of the same kind which are proportionals,
so that $A : B :: C : D :: E : F :: \text{etc.},$
then $A : B :: A+C+E+\text{etc.} : B+D+F+ \text{ etc.}$
(This inference is referred to as *addendo.*)

Because $A : B :: C : D :: E : F ::$ etc.
according as $mA >=$ or $< nB$,
so is $mC >=$ or $< nD$, *Def.* 4.
and $mE >=$ or $< nF$, etc. *Def.* 4.
and therefore so is $mA+mC+mE+\ldots >=$ or $< nB+nD+nF+\ldots$
or $m(A+C+E+\ldots) >=$ or $< n(B+D+F+\ldots)$
Props. of Mults. 3.

therefore, m and n being any numbers whatever,
$A : B :: A+C+E+\ldots : B+D+F+\ldots$ *Def.* 4.
Q.E.D.

Ex. 4. Prove also that
(1) $A : B :: A-C : B-D$;
(2) $A : B :: mA \pm nC : mB \pm nD$;
(3) $A : B :: mA+nB+pC+\ldots : mB+nD+pF+\ldots$

THEOR. 10. **If two ratios are equal, the sum or difference of the antecedent and consequent of the first has to the consequent the same ratio as the sum or difference of the antecedent and consequent of the other has to its consequent.**

Let $A : B :: P : Q$,
then $A+B : B :: P+Q : Q$ and $A \sim B : B :: P \sim Q : Q$.
(These inferences are referred to as *componendo* and *dividendo* respectively.)

If m be any number, n may be found such that mA is between nB and $(n+1)B$ or equal to nB,
therefore $mA+mB$ is between $mB+nB$ and $mB+(n+1)B$ or is equal to $mB+nB$:

but $mA+mB=m(A+B)$, and $mB+nB=(m+n)B$,

Props. of Mults., 3 *and* 5.

therefore $m(A+B)$ is between $(m+n)B$ and $(m+n+1)B$ or equal to $(m+n)B$.

Also because $A : B :: P : Q$,

therefore mP is between nQ and $(n+1)Q$ or equal to nQ,

Def. 4.

and therefore by similar reasoning

$m(P+Q)$ is between $(m+n)Q$ and $(m+n+1)Q$ or equal to $(m+n)Q$,

so that the multiples of $P+Q$ lie among those of Q in the same way as the multiples of $A+B$ among those of B,

therefore $A+B : B :: P+Q : Q$. *Def.* 4.

A similar proof, *mutatis mutandis* and applying the properties of multiples 4 and 6, will show that

$$A-B : B :: P-Q : Q, \text{ if } A > B;$$

or,

$$B-A : B :: Q-P : Q, \text{ if } A < B.$$

Q.E.D.

Ex. 5. Prove the converse of the first part, and thence deduce the second part.

Ex. 6. If $A : B > P : Q$, then $A+B : B > P+Q : Q$; and $A \sim B : B > \text{or} < P \sim Q : Q$, as A is $>$ or $<$ B.

THEOR. 11. **If two ratios are equal, and equimultiples of the antecedents and also of the consequents are taken, the multiple of the first antecedent has to that of its consequent the same ratio as the multiple of the other antecedent has to that of its consequent.**

Let $A : B :: P : Q$;

then $mA : nB :: mP : nQ$.

Let p, q be any two numbers,
then because $A : B :: P : Q$,
according as $pm.A > = \text{ or } < qn.B$,
so is $pm.P > = \text{ or } < qn.Q$, *Def.* 4.
that is, as $p.mA > = \text{ or } < q.nB$,
so is $p.mP > = \text{ or } < q.nQ$;
whence, since p and q are any numbers,
$mA : nB :: mP : nQ$. *Def.* 4.
Q.E.D.

THEOR. 12. **If there are two sets of magnitudes, such that the first is to the second of the first set as the first to the second of the other set, and the second to the third of the first set as the second to the third of the other, and so on to the last magnitude: then the first is to the last of the first set as the first to the last of the other.**

First let there be three magnitudes, A,B,C, of one set and three, P,Q,R, of another set,
such that $A : B :: P : Q$, and $B : C :: Q : R$;
then $A : C :: P : R$.
(This inference is referred to as *ex æquali.*)
Because $mA : mB :: A : B$, *IV.* 5.
and $mP : mQ :: P : Q$, *IV.* 5.
therefore, since $A : B :: P : Q$, $mA : mB :: mP : mQ$. *IV.* 1.
Again, because $B : C :: Q : R$,
therefore $mB : nC :: mQ : nR$. *IV.* 11.
Now, if $mA > nC$, $mA : mB > nC : mB$; *IV.* 4.
whence, since $mA : mB :: mP : mQ$,
and $nC : mB :: nR : mQ$, *Invertendo.*

$$mP : mQ > nR : mQ,$$

and therefore $mP > nR$. *IV.* 4, *Cor.*

And similarly it may be shewn that if $mA = nC$, $mP = nR$,
and if $mA < nC$, $mP <$
therefore, m and n being any two numbers,

$$A : C :: P : R.$$ *Def.* 4.

Next, let there be any number of magnitudes A,B,C, . . . H, K of one set, and the same number P,Q,R . . . X,Y of another set, such that

$$A : B :: P : Q,\ B : C :: Q : R, \ldots\ldots,\ H : K :: X : Y;$$

then $A : K :: P : Y$.
For it has been proved that $A : C :: P : R$;
therefore, because $C : D :: R : S$,
for the like reason $A : D :: P : S$,
and so on continually,
so that finally $A : K :: P : Y$.

Q.E.D.

COR. **If** $A : B :: Q : R$ **and** $B : C :: P : Q$,
then $A : C :: P : R$.

Let S be the fourth proportional to P, Q, R,
so that $P : Q :: R : S$;
then $A : B :: Q : R$, and $B : C :: R : S$; *IV.* 1.
therefore $A : C :: Q : S$; *Ex æquali.*
but because $P : Q :: R : S$,
therefore $P : R :: Q : S$, *Alternando.*
and therefore $A : C :: P : R$. *IV.* 1.

Q.E.D.

* *Ex.* 7. Prove that, if $A : B :: P : Q$,

$$A+B : A\sim B :: P+Q : P\sim Q.$$

THEOR. 13. **If A : C : : P : R, and B : C : : Q : R, then** A+B : C : : P+Q : R.

Because	B : C : : Q : R,	*Hyp.*
therefore	C : B : : R : Q,	*Invertendo.*
and	A : C : : P : R,	*Hyp.*
therefore	A : B : : P : Q,	*Ex æquali.*
therefore	A+B : B : : P+Q : Q,	*Componendo.*
and	B : C : : Q : R	
therefore	A+B : C : : P+Q : R.	*Ex æquali.*

Q.E.D.

DEF. 9. *If there are any number of magnitudes of the same kind, the first is said to have to the last the ratio* **compounded** of *the ratios of the first to the second, of the second to the third, and so on to the last magnitude.*

Thus A : K is the ratio compounded of A : B and B : K, or of A : B, B : C, and C : K, or of A : B, B : C, C : D and D : K, and so on.

DEF. 10. *If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set is said to have to the last the ratio* **compounded** of *the original ratios.*

Thus, if A : B : : P : Q and C : D : : Q : R, the ratio P : R is that compounded of A : B and C : D.
If also E : F : : R : S, the ratio P : S is that compounded of
A : B, C : D, and : E : F.

Hence Theor. 12 may be thus expressed:

If there are two sets of ratios equal to one another, each to each, the ratio compounded of the ratios of the first set is equal to that compounded of the ratios of the other set.

DEF. 11. *When two ratios are equal, the ratio compounded of them is called the* **duplicate** *ratio of either of the original ratios.*

DEF. 12. *When three ratios are equal, the ratio compounded of them is called the* **triplicate** *ratio of any one of the original ratios.*

THEOR. 14. **If two ratios are equal, their duplicate ratios are equal; and conversely, if the duplicate ratios of two ratios are equal, the ratios themselves are equal.**

Let A : B : : P : Q,
then the duplicate ratio of A : B is equal to that of P : Q.

Let C, R be the third proportionals to A, B and to P, Q
respectively, so that A : B : : B : C,
and P : Q : : Q : R;
then because A : B : : P : Q,
therefore B : C : : Q : R; *IV.* 1.
whence, *ex æquali,* A : C : : P : R,
that is, the duplicate ratio of A : B is equal to that of P : Q.

Next let the duplicate ratio of A : B be equal to that of P : Q,
that is, let A : C : : P : R.

Suppose S to be such that A : B : : P : S,
therefore B : A : : S : P *Invertendo.*
and A : C : : P : R;

whence, *ex æquali*, B : C : : S : R;

and, since A : B : : B : C, A : B : : S : R, *IV.* 1.

but A : B : : P : S;

therefore P : S : : S : R, *IV.* 1.

or S is the mean proportional between P and R,

so that S = Q,

hence A : B : : P : Q.

Q.E.D.

Ex. 8. The duplicate ratio of $m : n$, m and n being two numbers, is $m^2 : n^2$.

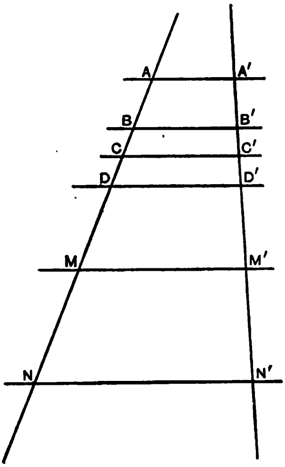
Ex. 9. The ratio compounded of $m : n$ and $p : q$, m, n, p, q, being numbers, is $mp : nq$.

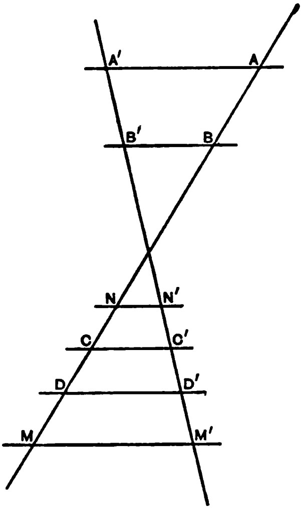
SECTION II.

FUNDAMENTAL GEOMETRICAL PROPOSITIONS.

THEOR. 15. **If two straight lines are cut by a series of parallel straight lines, the intercepts on the one have to one another the same ratios as the corresponding intercepts on the other have.**

Let a series of parallel lines cut one line in the points A, B, C, D . . . and another in the points A′, B′, C′, D′, etc., so that AB and A′B′ are corresponding intercepts, as also CD, C′D′:





then shall AB be to CD as $A'B'$ to $C'D'$.

On the line ABC take $AM = m.AB$ and $AN = n.CD$, M and N being on the same side of A,
and draw MM', NN' parallel to any line of the series,
and therefore parallel to all and to one another. *I.* 24.
Then if AM were divided into m parts, each equal to AB, and lines were drawn through the points of division parallel to AA' or MM', the corresponding intercepts (m in number) would be each equal to $A'B'$, *I.* 34.
and therefore $A'M' = m.A'B'$.
In like manner $A'N' = n.C'D'$.
Then according as MM' lies on the same side of NN' as AA', coincides with NN', or lies on the other side of NN',
so is $AM <= \text{ or } > AN$,
and $A'M' <= \text{ or } > A'N'$,
that is, as $m.AB <= \text{ or } > n.CD$,
so is $m.A'B' <= \text{ or } > n.C'D'$,
therefore, m and n being any two numbers,
$AB : CD :: A'B' : C'D'$. *Def.* 4.

Q.E.D.

THEOR. 16. **A given finite straight line can be divided internally into segments having any given ratio, and also externally into segments having any given ratio except the ratio of equality; and in each case there is only one such point of division.**

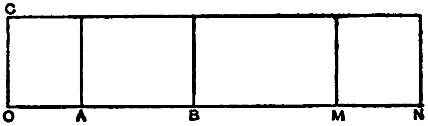
For demonstration, See Part I., Theor. 10.

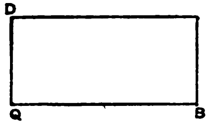
THEOR. 17. **A straight line which divides the sides of a triangle proportionally is parallel to the base of the triangle.**

For demonstration, See Part I., Theor. 11.

THEOR. 18. **Rectangles of equal altitude are to one another in the same ratio as their bases.**

Let AC, BD be two rectangles of equal altitude on the bases AO, BQ,





then shall the rectangle AC be to the rectangle BD as the base AO is to the base BQ.

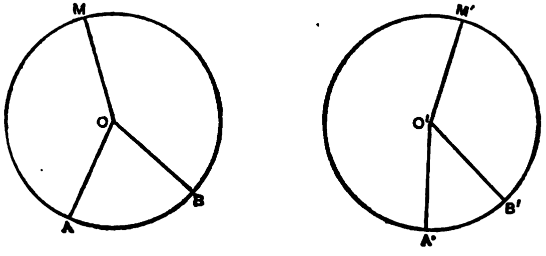
Since the rectangles are of equal altitude, QD is equal to OC, and if DQ be placed on CO, D upon C, and therefore Q on O, and B on the same side of CO as A, Q B will fall along OA.
Take $OM = m.OA$ and $ON = n.OB$,
and complete the rectangles MC, NC.
Then if OM were divided into m parts, each equal to OA, and rectangles of altitudes equal to OC described on each part, these rectangles would be all equal, *II.* 1, *Cor.* 2.
and they would together make up the rectangle MC,
therefore the rectangle $MC = m.$ the rectangle AC.
In like manner the rectangle $NC = n.$ the rectangle BC.
Then according as the base $OM >= \text{or} < ON$,
so is the rectangle $MC >= \text{or} < NC$; *II.* 1, *Cor.* 2.

that is, according as $m.\mathrm{OA} > = \text{ or } < n.\mathrm{OB}$,
so is m. the rectangle CA $> = \text{ or } <$ n. the rectangle CB;
therefore, m and n being any two numbers,
the rectangle CA : the rectangle CB : : base OA : base OB; *Def.* 4.
or the rectangle AC : the rectangle BD : : base AO : base BQ.

Q.E.D.

THEOR. 19. **In the same circle or in equal circles angles at the centre and sectors are to one another as the arcs on which they stand.**

Let O, O′ be the centres of two equal circles; AB, A′B′ any two arcs of these circles included between the radii OA, OB and O′A′, O′B′ respectively:
then, if the arc AB be that described by a point which moves on the circumference in any manner from A to B, and the angle AOB be that turned through by a radius drawn to the point, and the arc A′B′ and angle A′O′B′ be similarly estimated,



the angle AOB : the angle A′O′B′ : : arc AB : arc A′B′, and the sector AOB : the sector A′O′B′ : : arc AB : arc A′B′.

Let AM be an arc $= m$. the arc AB,

then, observing that Theors. 4 and 5 of Book III. are true for arcs of unlimited length (not necessarily less than one circumference), provided that the corresponding angles are reckoned as those turned through by a radius drawn to a point which moves along the whole arc from one extremity to the other (and therefore not necessarily less than two straight angles),

because the arc AM is made up of m arcs, each equal to AB,
and on each of these arcs stands an angle equal to AOB, *III.* 5.
therefore the angle AOM = m. the angle AOB,
and the sector AOM = m. the sector AOB, *III.* 4, *Cor.*

In like manner if A'M' = n.A'B',
the angle A'O'M' = n. the angle A'O'B',
and the sector A'O'M' = n. the sector A'O'B'.

Then according as the arc AM $>$ = or $<$ the arc A'M',
so is the angle AOM $>$ = or $<$ the angle A'O'M', *III.* 5.
and the sector AOM $>$ = or $<$ the sector A'O'M', *III.* 4, *Cor.*
that is, according as m.AB $>$ = or $<$ n.A'B',
so is m. the angle AOB $>$ = or $<$ n. the angle A'O'B',
and m. the sector AOB $>$ = or $<$ n. the sector A'O'B'.

Whence the angle AOB : the angle A'OB' : : arc AB : arc A'B',
and the sector AOB : the sector A'OB' : : arc AB : arc A'B', *Def.* 4.

Q.E.D.

DEFINITIONS OF BOOK IV.

PART II.

DEF. 1. *One magnitude is said to be a* **multiple** *of another magnitude, when the former contains the latter an exact number of times. According as the number of times is* 1, 2, 3 . . . *m, so is the multiple said to be the* 1*st*, 2*nd*, 3*rd* . . . *mth.*

DEF. 2. *One magnitude is said to be a* **measure** *or* **part** *of another magnitude, when the former is contained an exact number of times in the latter.*

DEF. 3. *The* **ratio** *of one magnitude to another of the same kind is the relation of the former to the latter in respect of* **quantuplicity**.

DEF. 4. *The ratio of one magnitude to another is equal to that of a third magnitude to a fourth (whether of the same or of a different kind from the first and second), when any equimultiples whatever of the antecedents of the ratios being taken and likewise any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as that of the other antecedent is greater than, equal to, or less than that of its consequent.*

DEF. 5. *The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when equimultiples of the antecedents and equimultiples of the consequents can be found such that, while the multiple of the antecedent of the first is greater than, or equal to, that of its consequent, the multiple of the antecedent of the other is not greater, or is less, than that of its consequent.*

DEF. 6. *When the ratio of A to B is equal to that of P to Q the four magnitudes are said to be* **proportionals** *or to form a* **proportion**. *The proportion is denoted thus:*

$$A : B :: P : Q,$$

then, observing that Theors. 4 and 5 of Book III. are true for arcs of unlimited length (not necessarily less than one circumference), provided that the corresponding angles are reckoned as those turned through by a radius drawn to a point which moves along the whole arc from one extremity to the other (and therefore not necessarily less than two straight angles),
because the arc AM is made up of m arcs, each equal to AB,
and on each of these arcs stands an angle equal to AOB, *III.* 5.
therefore the angle AOM $=$ m. the angle AOB,
and the sector AOM $=$ m. the sector AOB, *III.* 4, *Cor.*

In like manner if A′M′ $=$ n.A′B′,
the angle A′O′M′ $=$ n. the angle A′O′B′,
and the sector A′O′M′ $=$ n. the sector A′O′B′.

Then according as the arc AM $> = $ or $<$ the arc A′M′,
so is the angle AOM $> =$ or $<$ the angle A′O′M′, *III.* 5.
and the sector AOM $> =$ or $<$ the sector A′O′M′, *III.* 4, *Cor.*
that is, according as m.AB $> =$ or $<$ n.A′B′,
so is m. the angle AOB $> =$ or $<$ n. the angle A′O′B′,
and m. the sector AOB $> =$ or $<$ n. the sector A′O′B′.

Whence the angle AOB : the angle A′OB′ : : arc AB : arc A′B′,
and the sector AOB : the sector A′OB′ : : arc AB : arc A′B′,

Def. 4.

Q.E.D.

DEFINITIONS OF BOOK IV.

PART II.

DEF. 1. *One magnitude is said to be a* **multiple** *of another magnitude, when the former contains the latter an exact number of times. According as the number of times is* 1, 2, 3 . . . *m, so is the multiple said to be the* 1*st*, 2*nd*, 3*rd* . . . *mth*.

DEF. 2. *One magnitude is said to be a* **measure** *or* **part** *of another magnitude, when the former is contained an exact number of times in the latter.*

DEF. 3. *The* **ratio** *of one magnitude to another of the same kind is the relation of the former to the latter in respect of* **quantuplicity**.

DEF. 4. *The ratio of one magnitude to another is equal to that of a third magnitude to a fourth (whether of the same or of a different kind from the first and second), when any equimultiples whatever of the antecedents of the ratios being taken and likewise any equimultiples whatever of the consequents, the multiple of one antecedent is greater than, equal to, or less than that of its consequent, according as that of the other antecedent is greater than, equal to, or less than that of its consequent.*

DEF. 5. *The ratio of one magnitude to another is greater than that of a third magnitude to a fourth, when equimultiples of the antecedents and equimultiples of the consequents can be found such that, while the multiple of the antecedent of the first is greater than, or equal to, that of its consequent, the multiple of the antecedent of the other is not greater, or is less, than that of its consequent.*

DEF. 6. *When the ratio of A to B is equal to that of P to Q the four magnitudes are said to be* **proportionals** *or to form a* **proportion**. *The proportion is denoted thus :*

$$A : B :: P : Q,$$

which is read "A is to B as P is to Q." A and Q are called the **extremes**, *B and P the* **means**, *and Q is said to be the* **fourth proportional** *to A, B, and P. The antecedents A, P are said to be* **homologous**, *and so are the consequents B, Q.*

DEF. 7. *Three magnitudes (A, B, C) of the same kind are said to be proportionals, when the ratio of the first to the second is equal to that of the second to the third; that is, when A : B : : B : C.*

In this case C is said to be the **third proportional** *to A and B, and B the* **mean proportional** *between A and C.*

DEF. 8. *The ratio of any magnitude to an equal magnitude is said to be a* **ratio of equality**. *If A be greater than B, the ratio A : B is said to be a* **ratio of greater inequality**, *and the ratio B : A a* **ratio of less inequality**. *Also the ratios A : B and B : A are said to be* **reciprocal** *to one another.*

DEF. 9. *If there are any number of magnitudes of the same kind, the first is said to have to the last the ratio* **compounded** *of the ratios of the first to the second, of the second to the third, and so on to the last magnitude.*

DEF. 10. *If there are any number of ratios, and a set of magnitudes is taken such that the ratio of the first to the second is equal to the first ratio, and the ratio of the second to the third is equal to the second ratio, and so on, then the first of the set is said to have to the last the ratio* **compounded** *of the original ratios.*

DEF. 11. *When two ratios are equal, the ratio compounded of them is called the* **duplicate** *ratio of either of the original ratios.*

DEF. 12. *When three ratios are equal, the ratio compounded of them is called the* **triplicate** *ratio of any one of the original ratios.*

BOOK V.

PROPORTION.

SECTION I.

SIMILAR FIGURES.

DEF. 1. *If the angles of a rectilineal figure, taken in order, are equal respectively to those of another, also taken in order, the figures are said to be* **equiangular.** *Each angle of the one is said to* **correspond** *to the angle equal to it in the other, and the sides joining the vertices of corresponding angles are termed* **corresponding** *sides.*

DEF. 2. **Similar** *figures are such as are equiangular, and have their sides proportionals, the corresponding sides being homologous.*

DEF. 3. *Similar figures are said to be* **similarly situated upon given straight lines,** *when those straight lines are corresponding sides of the figures.*

THEOR. 1. **Rectilineal figures that are similar to the same rectilineal figure are similar to one another.**

Let each of the rectilineal figures A and B be similar to the rectilineal figure C:
then shall A and B be similar to one another.

Because A is similar to C,
therefore the angles of A taken in order are equal to the angles of C also taken in order, each to each;
in like manner the angles of B taken in order are equal to the angles of C taken in order, each to each;
but things that are equal to the same thing are equal to one another,
therefore the angles of A taken in order are equal to the angles of B taken in order, each to each.

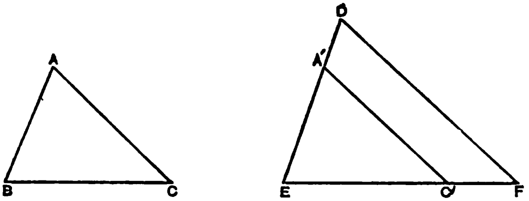
Again, because A is similar to C,
therefore the ratio of any two sides of A is equal to the ratio of the corresponding sides of C;
in like manner the ratio of any two sides of B is equal to the ratio of the corresponding sides of C;
but ratios that are equal to the same ratio are equal to one another,
therefore the corresponding sides of A and B are proportional.

Hence the figures A and B are similar to one another.

Q.E.D.

THEOR. 2. **If two triangles have their angles respectively equal, they are similar, and those sides which are opposite to the equal angles are homologous.**

Let ABC, DEF be two triangles having the angles at A, B respectively equal to the angles at D,E, and consequently (*I.* 25) the angle at C equal to the angle at F:



then shall the triangles be similar, having AB to BC as DE to EF, BC to CA as EF to FD, and CA to AB as FD to DE.

Apply the triangle ABC to the triangle DEF, so that B may fall on E and BA along ED, then BC will fall along EF, since the angle ABC is equal to the angle DEF.

Let A and C fall at A′ and C′ in ED and EF, or in these sides produced.

Then, because the angle EA′C′ is equal to the angle EDF,

therefore A′C′ is parallel to DF,

and therefore A′E : DE : : EC′ : EF, **IV.* 9, *Cor.* 2.

that is, AB : DE : : BC : EF,

and therefore, *alternando*, AB : BC : : DE : EF.

In like manner it may be shown that

BC : CA : : EF : FD, and CA : AB : : FD : DE.

Q.E.D.

* *Ex.* 1. If two straight lines be drawn from the same point A, one touching a circle in B, and the other cutting it in C and D, the triangles ABC, ABD are similar.

Ex. 2. ABC is an acute-angled triangle, and AD, BE are the perpendiculars from A and B on the opposite sides. Shew that the triangle CDE is similar to the triangle ABC.

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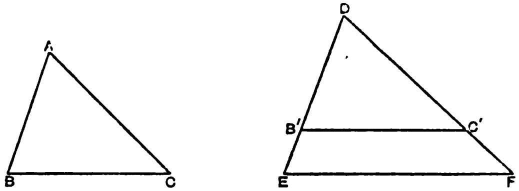
* The references are to Part I. of Book IV.

Ex. 3. With centre B and radius BC a circle is described, cutting AC, one of the equal sides AB, AC of an isosceles triangle, in D. Prove that BC is a mean proportional between AC, CD.

Ex. 4. Two equal triangles are drawn upon the same base, and a straight line is drawn through them parallel to the base: shew that the parts of it intercepted between the sides of each triangle are equal to one another.

THEOR. 3. **If two triangles have one angle of the one equal to one angle of the other and the sides about these angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal.**

Let ABC, DEF be two triangles having the angle BAC equal to the angle EDF, and the side AB to the side AC as the side DE to the side DF:



then shall the triangles be similar, having the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.

Apply the triangle ABC to the triangle DEF, so that A may fall on D, and AB along DE then AC will fall along DF, since the angle BAC is equal to the angle EDF.

Let B and C fall at B′ and C′ in DE and DF, or in these sides produced.

Then, because AB : AC : : DE : DF,

therefore DB′ : DE : : DC′ : DF, *Alternando.*

therefore B′C′ is parallel to EF, *IV.* 11.

and therefore the angle DB′C′ is equal to the angle DEF, and the angle DC′B′ to the angle DFE, that is the angle ABC is equal to the angle DEF, and the angle ACB to the angle DFE.

Thus the triangles ABC, DEF are equiangular, and therefore by Theor. 2 are similar.

Q.E.D.

Ex. 5. If the perpendicular drawn from the vertex of a triangle to the base fall within the triangle and is a mean proportional between the segments of the base, the triangle is right-angled.

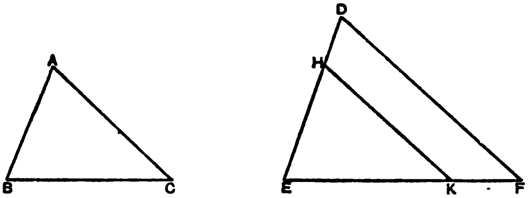
Ex. 6. If APB is a semicircle, of which AB is the diameter and C the centre, N a point on CB, and AB is produced to T, so that CT : AC : : AC : CN, and PT is the tangent drawn from T, then CNP is a right angle.

Ex. 7. If AB, CD, produced if necessary, intersect at E, and AE : EC : : ED : EB, the points A, B, C, D are concyclic.

Ex. 8. If two quadrilaterals have three angles of the one equal to three angles of the other, and the sides about one pair of angles proportional, the sides adjacent to equal angles being homologous terms of the proportion, then the quadrilaterals are similar.

THEOR. 4. **If two triangles have the sides taken in order about each of their angles proportional, they are similar, and those angles which are opposite to the homologous sides are equal.**

Let ABC, DEF be two triangles having the side AB to the side BC as the side DE to the side EF, the side BC to the side CA as the side EF to the side FD, and consequently, *ex æquali*, the side AB to the side AC as the side DE to the side DF:



then shall the angle ABC be equal to the angle DEF, the angle BCA to the angle EFD, and the angle CAB to the angle FDE.

In ED make EH equal to BA, and in EF make EK equal to BC, join HK.

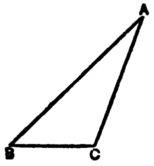
Then, because AB : BC : : DE : EF, *Hyp.*
therefore HE : EK : : DE : EF,
and the angle DEF is common to the triangles HEK, DEF,
therefore the triangles are similar, *V.* 3.
and therefore EK : KH : : EF : FD;
but BC : CA : : EF : FD, *Hyp.*
therefore EK : KH : : BC : CA,
but EK is equal to BC,

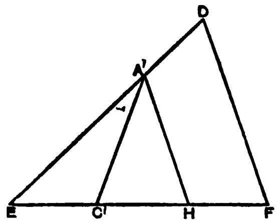
therefore KH is equal to CA. *IV.* 2.
Hence, in the triangles ABC, HEK,
the side AB is equal to the side HE,
the side BC is equal to the side EK,
and the side CA is equal to the side KH,
therefore the triangles are identically equal. *I.* 18.
But the triangle HEK has been proved similar to the triangle DEF,
therefore the triangle ABC is similar to the triangle DEF, having the angles at A, B, C respectively equal to the angles at D, E, F. *V.* 1.

Q.E.D.

THEOR. 5. **If two triangles have one angle of the one equal to one angle of the other, and the sides about one other angle in each proportional, so that the sides opposite the equal angles are homologous, the triangles have their third angles equal or supplementary, and in the former case the triangles are similar.**

Let ABC, DEF be two triangles having the angle ABC equal to the angle DEF, and the side AB to the side AC as the side DE to the side DF:





then shall the angles ACB, DFE be either equal or supplementary, and, in the former case, the triangles shall be similar.

Apply the triangle ABC to the triangle DEF, so that B may fall on E, and BA along ED, then BC will fall along EF,
since the angle ABC is equal to the angle DEF.
Let A and C fall at A′ and C′ in ED and EF, or in these sides produced.

Then, if A′C′ is parallel to DF,
the angle A′C′E is equal to the angle DFE,
that is, the angle ACB is equal to the angle DFE,
and therefore the triangles ABC and DEF are similar. *V.* 2.

But if A′C′ is not parallel to DF,
from A′ draw A′H parallel to DF, and meeting EF in H.
Then the triangles A′EH, DEF are equiangular,
therefore EA′ : A′H : : ED : DF, *V.* 2.
but EA′ : A′C′ : : ED : DF, *Hyp.*
therefore EA′ : A′H : : EA′ : A′C′,
and therefore A′H is equal to A′C′, *IV.* 2.
therefore the angle A′C′H is equal to the angle A′HC′,
therefore the angles A′C′E, A′HC′ are supplementary,
and therefore the angles ACB, DFE are supplementary.

Q.E.D.

Cor. Two such triangles are similar—

1. If the two angles given equal are right angles or obtuse angles.

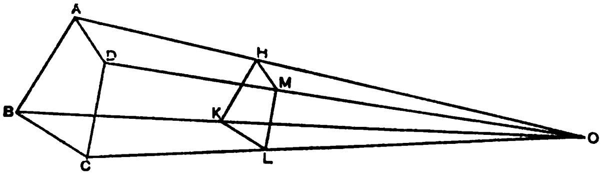
2. If the angles opposite to the other two homologous sides are both acute or both obtuse, or if one of them is a right angle.

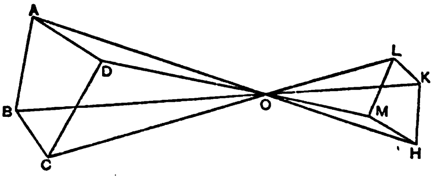
3. If the side opposite the given angle in each triangle is not less than the other given side.

Ex. 9. If OMN, OPQ are two straight lines, and MP, NQ meet in R, then if OM : MP :: ON : NQ, the triangle PQR is isosceles.

THEOR. 6. **If two similar rectilineal figures are placed so as to have their homologous sides parallel, all the straight lines joining the angular points of the one to the corresponding angular points of the other are parallel or meet in a point; and the distances from that point along any straight line to the points where it meets homologous sides of the figures are in the ratio of the homologous sides of the figures.**

Let ABCD, HKLM be two similar rectilineal figures having the sides AB, BC, CD, DA respectively parallel to the homologous sides HK, KL, LM, MH:





then shall the straight lines AH, BK, CL, DM be all parallel or meet in a point.

First, let the lines AB, HK be drawn in the same sense (*i.e.*, in the same direction, or towards the same parts) from A and H.

Then, if the figures are equal as well as similar, ABKH is a quadrilateral having its opposite sides AB, HK equal and parallel,
therefore AH is parallel to BK; *I.* 31.
and in like manner BK is parallel to CL,
and CL to DM.

But if the figures are not equal,
let AH be produced to cut BK externally in O.
Then because AB and HK are parallel,
therefore the triangles ABO, HKO are equiangular,
and therefore BO : KO : : AB : HK, *V.* 2.
that is, AH cuts BK externally in the ratio AB : HK.
In like manner CL cuts BK externally in the ratio BC : KL,
but AB : HK : : BC : KL, since the figures are similar,
therefore AH and CL cut BK externally in the same ratio,
and therefore in the same point O. *IV.* 10.

By like reasoning, DM passes through the point O in which BK cuts CL.

Hence AH, BK, CL, DM meet in the same point O.

Secondly, let the lines AB, HK be drawn in contrary senses from A and H.

Then, by like reasoning, it may be proved that AH, BK, CL, DM cut one another internally in the same point O in the ratio of homologous sides of the figures.

In this case the proof holds when the figures are equal, but we may then obtain the result more simply from the theorem that the diagonals of a parallelogram bisect one another.

Again, let any straight line be drawn through O to meet corresponding sides AB, HK in P, Q respectively:
then shall OP be to OQ as AB is to HK.

Because AP and HQ are parallel,
therefore the triangles APO, HQO are equiangular,
and therefore OP : OQ : : OA : OH; *V.* 2.
but OA : OH : : AB : HK,
therefore OP : OQ : : AB : HK.

Q.E.D.

Cor. **Similar rectilineal figures may be divided into the same number of similar triangles.**

For if the figures be placed so that their corresponding sides are parallel, and one figure lies wholly within the other, the lines joining corresponding angular points will meet in a point within the figures, and will divide the figures into as many pairs of similar triangles as each figure has sides.

Def. 4. *The point of intersection of all straight lines which join the corresponding points of two similar figures, whose corresponding sides are parallel, is called the* **centre of similarity** *of the two figures.*

Ex. 10. If any point O is joined to the angular points A, B, C . . . of a rectilineal figure, and points *a*, *b*, *c* . . . are taken in OA, OB, OC . . . such that O*a* : OA : : O*b* : OB : : O*c* : OC, etc., then the figure *abc* . . . is similar to the figure ABC . . .

Ex. 10*a*. If ABCD . . ., *abcd* . . . are two figures such that the lines A*a*, B*b*, C*c* . . . meet in the same point O, then, if the sides AB, BC . . . are respectively parallel to *ab*, *bc* . . ., the figures are similar.

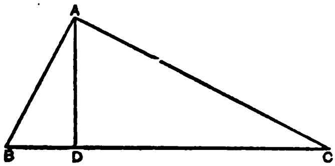
Is it true that, if the figures are similar, the sides are parallel?

Ex. 11. O is a fixed point, P any point on the circumference of a given circle. Q is taken on OP such that OQ : OP is a fixed ratio. Prove that Q lies on the circumference of a circle.

Ex. 12. O is a fixed point, P any point on the circumference of a given circle. OQ is taken making a given angle with OP, and having a fixed ratio to it. Shew that Q lies on the circumference of one of two circles.

THEOR. 7. **In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, it divides the triangle into two other triangles which are similar to the whole and to one another.**

Let ABC be a triangle having the angle BAC a right angle, and let AD be perpendicular to BC:



then shall the triangles DBA, DAC be similar to the triangle ABC and to one another.

In the triangles DBA, ABC,
the angle ABC is common,
and the right angles BDA, BAC are equal,
therefore the triangles are similar. *V.* 2.
In like manner the triangles DAC, ABC are similar.

Hence the triangles DBA, DAC being similar to the same triangle ABC are also similar to one another. *V.* 1.

Q.E.D.

COR. **Each side of the triangle is a mean proportional between the hypotenuse and the adjacent segment of the hypotenuse; and the perpendicular is a mean proportional between the segments of the hypotenuse.**

Ex. 13. Name the sides of the three triangles which are homologous to one another.

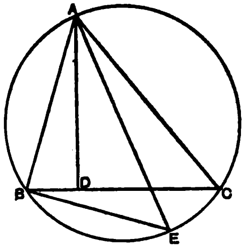
Ex. 14. The segments of the base of the triangle are in the duplicate ratio of the sides of the triangle.

Ex. 15. Shew that the radius of a circle is a mean proportional between the segments of any tangent between the point of contact and any two parallel tangents.

Ex. 16. If two circles touch externally, the part of each of the two symmetrical common tangents between its points of contact is a mean proportional between the diameters.

THEOR. 8. **If from any angle of a triangle a straight line is drawn perpendicular to the base, the diameter of the circle circumscribing the triangle is the fourth proportional to the perpendicular and the sides of the triangle which contain that angle.**

Let AE be a diameter of the circle circumscribing the triangle ABC, AD the perpendicular on BC:



then shall AD be to AC as AB to AE.

Join BE.

Because ABE is a semicircle,
therefore the angle ABE is a right angle, *III.* 17.
therefore the angle ADC is equal to the angle ABE.
Because the angles ACD and AEB are in the same segment ACB,
therefore the angle ACD is equal to the angle AEB. *III.* 16.
Hence the triangles ACD, AEB are equiangular,
and therefore AD : AC : : AB : AE. *V.* 2.

Q.E.D.

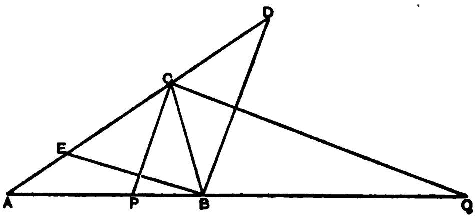
Ex. 17. D is a point on the base BC, or base produced, of the isosceles triangle ABC : shew that the circumscribed circles of the triangles ABD, ACD are equal.

Ex. 18. D is any point on the side BC of the triangle ABC: shew that the diameters of the circles about the triangles ABD, ACD are in the ratio of AB to AC.

THEOR. 9. **If the interior or exterior vertical angle of a triangle is bisected by a straight line which also cuts the base, the base is divided internally or externally in the ratio of the sides of the triangle.**

And, conversely, if the base is divided internally or externally in the ratio of the sides of the triangle, the straight line drawn from the point of division to the vertex bisects the interior or exterior vertical angle.

Let ABC be a triangle, and let CP and CQ, the bisectors of the interior and exterior vertical angles at C, cut the base AB in P and Q :



then shall AB be divided internally at P and externally at Q in the ratio AC : CB.

From B draw BD parallel to CP, and BE parallel to CQ to meet AC at D and E.

Then because CP is parallel to BD,
therefore the angle CBD is equal to the angle BCP,
and the angle CDB to the angle ACP;
but the angle BCP is equal to the angle ACP, *Hyp.*
therefore the angle CBD is equal to the angle CDB,
and therefore CD is equal to CB.

Again, because PC is parallel to BD,
therefore AP : PB : : AC : CD : *IV.* 9, *Cor.* 2.
but CD is equal to CB,
therefore AP : PB : : AC : CB.

In like manner it may be shewn that
AQ : QB : : AC : CB.

Conversely, let the base AB of the triangle ABC be divided internally at P and externally at Q
in the ratio AC : CB.
then shall CP and CQ be the bisectors of the interior and exterior vertical angles at C.

For the bisectors have been shewn to divide AB internally and externally in the ratio AC : CB,
and there is only one point in which AB can be divided internally, and only one point in which it can be divided externally in the ratio AC : CB, *IV.* 10.
and P, Q are these points of division, *Hyp.*
therefore, by the Rule of Identity, CP bisects the vertical angle ACB, and CQ the exterior vertical angle BCD.

Q.E.D.

Ex. 19. If two triangles have one angle of the one equal to one angle of the other, and a second angle of the one supplementary to a second angle of the other, then the sides about the third angles are proportional.

Ex. 20. ABC is a triangle which has its base BC bisected in D. DE, DF bisect the angles ADC, ADB, meeting AC, AB in E, F. Prove that EF is parallel to BC.

**Ex.* 21. OA, OB are two tangents to a circle whose centre is C; OPQ any straight line through O cutting the circle in P and Q, and AB in R. If N be the intersection of AB and OC, prove that NR bisects the angle PNQ, and consequently that PQ is divided harmonically at O and R.

**Ex.* 22. O is a fixed point, OPQ any straight line through O, cutting a given circle in P and Q; R is taken on PQ such that OQ : OP : : RQ : PR. Shew that R is on a fixed straight line.

EXERCISES.

*23. If three straight lines drawn from a point cut two parallel straight lines in A,B,C and A′,B′,C′ respectively, prove that AB : BC : : A′B′ : B′C′.

*24. If two parallel straight lines AB, A′B′ are divided both internally, or both externally, in the same ratio at C and C′ respectively, prove that AA′, BB′, CC′ meet in a point.

*25. The straight line DEF meets the sides BC, CA, AB at the points D,E,F respectively, and is equally inclined to AB and AC. Prove that BD : CD : : BF : CE.

26. D is a point in the side AC of a triangle ABC, E is a point in AB. If BD,CE divide each other into parts in the ratio 4 : 1, then D,E divide CA, BA in the ratio 3 : 1.

27. If AD bisect the angle BAC and meet BC in D, and DE, DF bisect the angles ADB, ADC, and meet AB, AC in

E, F respectively, prove that the triangle BEF is to the triangle CEF as BA is to AC.

28. If two isosceles triangles have equal vertical angles, their altitudes are to one another as their bases.

*29. Parallelograms about the diagonal of a parallelogram are similar to the whole, and to one another.

30. Similar and similarly placed parallelograms, which have a common angle, have their diagonals in the same straight line.

31. AEKH, KFCG are parallelograms about the diagonal AC of a parallelogram. Prove that EF, AC, GH are either parallel or meet in a point.

32. If ABC be a triangle inscribed in a circle, and the tangent AD at A meet BC produced in D, the diameters of the circles about ABD, ACD are as AD to CD.

33. Two circles touch internally at O. A straight line touches the inner circle at C, and meets the outer in A,B; and OA,OB meet the inner circle in P,Q. Shew that

$$OP : OQ :: AC : CB.$$

34. A series of triangles is constructed such that the straight lines drawn from the vertices to the middle points of the opposite sides of each are equal to the sides of the next. Prove that the alternate triangles of the series are similar.

35. One of the angles B of a right-angled isosceles triangle ABC is trisected by lines which meet the straight line AMN drawn from A the right angle perpendicular to the base BC in M and N, and CN produced cuts AB in E. Shew that EM is parallel to BN.

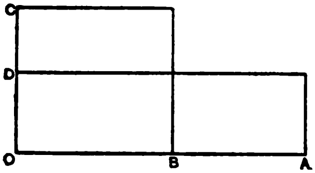
36. If three similar rectilineal figures be so placed as to have their corresponding sides parallel, their three centres of similarity, when taken two and two, lie in a straight line.

SECTION II.

AREAS.

THEOR. 10. **If four straight lines are proportional the rectangle contained by the extremes is equal to the rectangle contained by the means;**
and, conversely, if the rectangle contained by the extremes is equal to the rectangle contained by the means the four straight lines are proportional.

Let OA, OB, OC, OD be four straight lines, such that OA : OB : : OC : OD :



then shall the rectangle contained by OA and OD be equal to the rectangle contained by OB and OC.

Let OA and OB be placed along one of the arms of a right angle, OC, OD along the other arm;
complete the rectangles AD, BC,
and let BD be the common part of AD, BC.

Because rectangles of equal altitude are to one another as their bases,
therefore the rectangle AD is to the rectangle BD as OA is to OB.
For the same reason, the rectangle BC is to the rectangle BD as OC is to OD;
but OA : OB : : OC : OD, *Hyp.*
therefore the rectangle AD is to the rectangle BD as the rectangle BC is to the rectangle BD,
therefore the rectangle AD is equal to the rectangle BC, *IV.* 2.
that is, the rectangle contained by the extremes OA, OD is equal to the rectangle contained by the means OB, OC.

Conversely let the rectangle contained by OA and OD be equal to the rectangle contained by OB and OC :
then shall OA be to OB as OC is to OD.

Let the same construction be made as before.

Then, because the rectangle AD is equal to the rectangle BC, *Hyp.*
therefore the rectangle AD is to the rectangle BD as the rectangle BC is to the rectangle BD. *IV.* 2.
But the rectangle AD is to the rectangle BD as the base OA is to the base OB. *IV.* 12.
And the rectangle BC is to the rectangle BD as the base OC is to the base OD, *IV.* 12.
therefore OA : OB : : OC : OD.

Q.E.D.

COR. **If three straight lines are proportional, the rectangle contained by the extremes is equal to the square on the mean;**

and, conversely, if the rectangle contained by the extremes of three straight lines is equal to the square on the mean the lines are proportional.

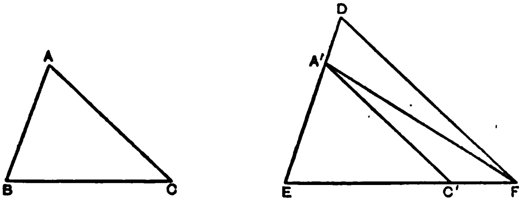
* *Ex.* 37. Apply Theor. 10 to shew that, if two chords of a circle intersect either within or without a circle, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

Ex. 38. If two chords AB, AC drawn from any point A in the circumference of a circle be produced to meet the tangent at the other extremity of the diameter through A in D and E, the triangles AED, ABC are similar.

Ex. 39. If ABC be a right-angled triangle, whose right angle B is bisected by BF, cutting the base in F, and meeting the circumference of the circle described about ABC in D, prove that the rectangle contained by BD and BF is equal to twice the area of ABC.

THEOR. 11. **Similar triangles are to one another in the duplicate ratio of their homologous sides.**

Let ABC, DEF be two similar triangles, having the sides BC, EF homologous:



then shall the triangle ABC be to the triangle DEF in the duplicate ratio of BC to EF.

Apply the triangle ABC to the triangle DEF so that B may fall on E and BC along EF,
then BA will fall along ED,
since the angle ABC is equal to the angle DEF. *Hyp.*
Let A′,C′ be the positions of A and C in ED, EF or in these sides produced.
Join A′F.

Because triangles of the same altitude are to one another as their bases,
therefore the triangle A′EC′ is to the triangle A′EF as EC′ is to EF, that is, as BC is to EF.
For the same reason, the triangle A′EF is to the triangle DEF as A′E is to DE, that is, as AB : DE;
but AB : BC : : DE : EF, *Hyp.*
therefore AB : DE : : BC : EF, *Alternando.*
and therefore the triangle A′EF is to the triangle DEF as BC : EF.

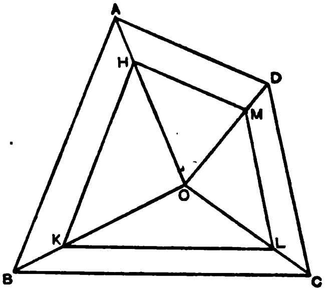
Now, the triangle A′EC′ has to the triangle DEF the ratio compounded of the ratios of the triangle A′EC′ to the triangle A′EF and of the triangle A′EF to the triangle DEF, *IV. Def.* 7.
and each of these ratios has been shewn to be equal to the ratio BC : EF,
therefore the triangle A′EC′, that is, the triangle ABC, has to the triangle DEF the duplicate ratio of BC to EF. *IV. Def.* 9.

Q.E.D.

Ex. 40. ABC is a triangle. AE and BF, intersecting in G, are drawn to bisect the sides BC, AC in E and F. Compare the areas of the triangles AGB, FGE.

THEOR. 12. **The areas of similar rectilineal figures are to one another in the duplicate ratio of their homologous sides.**

Let ABCD, HKLM be two similar figures, having the sides AB, HK homologous:



then shall the figure ABCD be to the figure HKLM in the duplicate ratio of AB, HK.

Let the figures be placed so as to have their homologous sides parallel, and so that one figure is within the other.
Let AH, BK, CL, DM be produced to meet in the centre of similarity O, *V.* 6.
then the figures are divided into as many pairs of similar triangles as each figure has sides, *V.* 6. *Cor.*
and the triangles of each pair are to one another in the duplicate ratio of their bases, *V.* 11.

that is, in the duplicate ratio of homologous sides of the figures, therefore the sum of the triangles that make up the one figure is to the sum of the triangles that make up the other in the duplicate ratio of homologous sides of the figures, *IV.* 6.
that is, the figure ABCD is to the figure HKLM in the duplicate ratio of AB to HK.

Q.E.D.

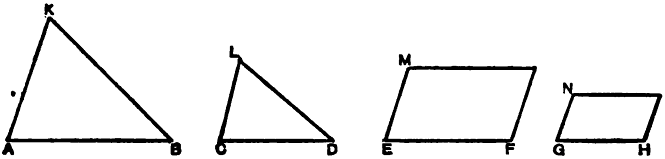
COR. **Similar rectilineal figures are to one another as the squares described on their homologous sides.**

For the figures and the squares are each to another in the duplicate ratio of the homologous sides.

Ex. 41. A square and a hexagon are inscribed in the same circle and equilateral triangles are described on their sides. Find the ratio of a triangle on the side of the square to one on the side of the hexagon.

THEOR. 13. **If four straight lines are proportional and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional;**
and conversely, if a rectilineal figure on the first of four straight lines is to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines are proportional.

Let AB, CD, EF, GH be four straight lines in proportion, and let KAB, LCD be similar and similarly described rectilineal figures on AB, CD, and also MF, NH similar and similarly described figures on EF, GH:



then shall the figure KAB be to the figure LCD as the figure MF is to the figure NH.

Because the ratio AB : CD is equal to the ratio EF : GH, therefore the duplicate ratio of AB : CD is equal to the duplicate ratio of EF : GH; *IV.* 14, *Part ii.*
but the figure KAB is to the figure LCD in the duplicate ratio of AB : CD, *V.* 12.
and the figure MF is to the figure NH in the duplicate ratio of EF : GH; *V.* 12.
therefore the figure KAB is to the figure LCD as the figure MF is to the figure NH.

Conversely, let a rectilineal figure KAB on AB be to LCD the similar and similarly described figure on CD as a rectilineal figure MF on EF is to the similar and similarly described figure NH on GH:
then shall AB be to CD as EF is to GH.

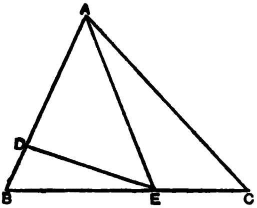
The figure KAB is to the figure LCD in the duplicate ratio of AB : CD, *V.* 12.
and the figure MF is to the figure NH in the duplicate ratio of EF : GH; *V.* 12.

but the figure KAB is to the figure LCB as the figure MF is to the figure NH, *Hyp.*
therefore the duplicate ratio of AB : CD is equal to the duplicate ratio of EF : GH.
Therefore AB : CD : : EF : GH. *IV.* 14, *Part ii.*

Q.E.D.

THEOR. 14. **If two triangles or parallelograms have one angle of the one equal to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.**

Let ABC, DBE be two triangles having the angle ABC equal to the angle DBE :



then shall the triangle ABC have to the triangle DBE the ratio compounded of the ratios BC : BE and BA : BD.

Apply the triangle ABC to the triangle DBE,
so that BC may fall along BE,
then BA will fall along BD,
since the angle ABC is equal to the angle DBE. *Hyp.*
Join AE.

Then the triangle ABC has to the triangle DBE the ratio compounded of the ratios of the triangle ABC to the triangle ABE, and of the triangle ABE to the triangle DBE.

But the ratio of the triangle ABC to the triangle ABE is equal to the ratio BC : BE, *IV.* 12, *Cor.*

and the ratio of the triangle ABE to the triangle DBE is equal to the ratio AB : DB. *IV.* 12, *Cor.*

Hence the ratio of the triangle ABC to the triangle DBE is equal to the ratio compounded of the ratios BC : BE and BA : BD.

A like proof may be given for parallelograms.

Q.E.D.

COR. 1. **If two triangles or parallelograms have one angle of the one supplementary to one angle of the other, their areas have to one another the ratio compounded of the ratios of the including sides of the first to the including sides of the second.**

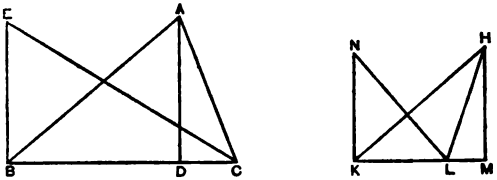
COR. 2. **The ratio compounded of two ratios between straight lines is the same as the ratio of the rectangle contained by the antecedents to the rectangle contained by the consequents.**

Ex. 42. D is any point in AC, the base of an isosceles triangle ABC. DE, DF make equal angles with AC, and meet the equal sides BC and AB in E and F. Prove that the triangles AED and CDF are equal.

Ex. 43. If ABC is a triangle right-angled at B, and BD the perpendicular on AC is produced to E, so that DE is a third proportional to BD and DC, the triangle ADE will be equal to the triangle BDC.

THEOR. 15. **Triangles and parallelograms have to one another the ratio compounded of the ratios of their bases and of their altitudes.**

Let ABC, HKL be two triangles, AD, HM their altitudes:



then shall the triangle ABC be to the triangle HKL in the ratio compounded of the ratios BC : KL and AD : HM.

From B draw BE at right angles to BC, and equal to AD, join EC;
and from K draw KN at right angles to KL and equal to HM, join NL.

Then the triangles ABC and EBC are equal,
and so also are the triangles HKL and NKL. *II.* 2. *Cor.* 1.
But because the angle EBC is equal to the angle NKL,
therefore the triangle EBC is to the triangle NKL in the ratio compounded of the ratios BC : KL and EB : NK, *V.* 14.
that is, of the ratios BC : KL and AD : HM.
Therefore the triangle ABC is to the triangle HKL in the ratio compounded of the ratios BC : KL and AD : HM.

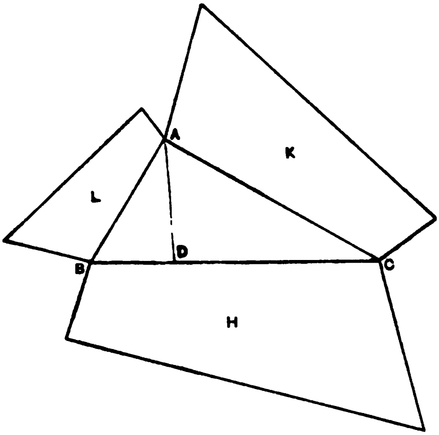
A like proof may be given for parallelograms.

Q.E.D.

Ex. 44. ABCD is a quadrilateral, having the side AB parallel to CD. The diagonals meet in O. Prove that the rectangle AO, OD is equal to the rectangle BO, OC.

THEOR. 16. **In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of two similar and similarly described figures on the sides.**

Let ABC be a triangle having the angle BAC a right angle, and let H, K, L be similar and similarly described rectilineal figures on BC, CA, AB:



then shall the figure H be equal to the sum of the figures K and L.

Draw AD perpendicular to BC.

Then BC : CA : : CA : CD, *V. 7. Cor.*

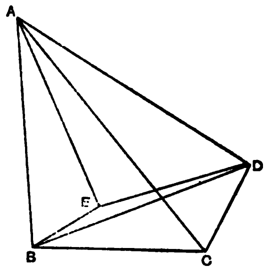
therefore BC : CD is the duplicate of the ratio BC : CA,
therefore the figure K is to the figure H as CD to BC. *V.* 12.
In like manner the figure L is to the figure H as BD to BC.
Therefore the sum of the figures K and L is to the figure H as the sum of CD and BD is to BC. *IV.* 8.
But the sum of CD and BD is equal to BC,
therefore the sum of the figures K and L is equal to the figure H.

Q.E.D.

Ex. 45. If these figures are rectangles, prove the proposition by a method analogous to that used in the first proof of II. 9.

THEOR. 17. **The rectangle contained by the diagonals of a quadrilateral is less than the sum of the rectangles contained by opposite sides unless a circle can be circumscribed about the quadrilateral, in which case it is equal to that sum.**

Let ABCD be a quadrilateral about which a circle cannot be described:



then shall the rectangle contained by AC, BD be less than the sum of the rectangles contained by AB, CD and by AD, BC.

Make the angle BAE equal to the angle CAD, and make the angle ABE equal to the angle ACD.
Join DE.

Then the triangles ABE, ACD are equiangular,
therefore BA : AE : : CA : AD, *V.* 2.
therefore BA : AC : : EA : AD; *Alternando.*
but the angle BAE is equal to the angle CAD,
therefore the whole angle BAC is equal to the whole angle EAD;
therefore the triangles BAC, EAD are similar. *V.* 3.

Because the triangles ABE, ACD are similar,
therefore AB : BE : : AC : CD,
therefore the rectangle contained by AB and CD is equal to the rectangle contained by AC and BE.
Again, because the triangles BAC, EAD are similar,
therefore BC : AC : : DE : AD,
therefore the rectangle contained by BC and AD is equal to the rectangle contained by AC and DE.
Hence the sum of the rectangles contained by AB and CD and by BC and AD is equal to the sum of the rectangles contained by AC and BE and by AC and DE.

But because BD is less than the sum BE and ED,
therefore the rectangle contained by AC and BD is less than the sum of the rectangles contained by AC and BE and by AC and DE,
therefore the rectangle contained by AC and BD is less than the sum of the rectangles contained by AB and CD and by BC and AD.

But if the quadrilateral ABCD be such that a circle can be described about it,

then the angle ABD is itself equal to the angle ACD,
therefore the point E lies on BD,
and therefore BD is equal to the sum of BE and ED,
therefore the rectangle contained by AC and BD is equal to the sum of the rectangles contained by AB and CD and by BC and AD.

Q.E.D.

Ex. 46. If the diagonals of a quadrilateral inscribed in a circle cut each other at right angles, shew that the rectangles contained by the opposite sides are together double of the quadrilateral figure.

EXERCISES.

47. If ACB, BCD be equal angles and DB be perpendicular to BC and BA to AC, prove that the triangle DBC is to the triangle ABC as DC is to CA.

48. ABCD is a square, P a point on the arc AB of its circumscribing circle. Shew that the rectangle PC, PD is equal to the sum of the rectangles PA, PB ; PB, PC ; PD, PA.

49. CA, CB are two radii of a circle at right angles to each other, DE is any chord. If BD, BE meet AC in F and G, prove that the triangles BFG, BDE are similar.

50. Similar triangles are in the ratio of the squares on the radii of their inscribed (or circumscribed) circles.

51. The sides of a regular hexagon are produced both ways, and the points of intersection joined to form a new regular hexagon. Prove that the area of the second hexagon is to that of the first in the ratio 3 : 1.

52. If from a point without a circle two straight lines

be drawn, one of which touches and the other cuts the circle, a line drawn from the same point in any direction equal to the tangent will be parallel to the chord of the arc intercepted by two lines drawn from its other extremity to the intersections of the circle with the cutting line.

53. If ABC be an equilateral triangle, and P a point on the circumscribing circle on the side of BC remote from A, prove that the square on PA is equal to the rectangle PB, PC together with the square on BC.

54. On the side BC of an acute-angled triangle ABC as diameter a circle is described. On AB a point D is taken such that AD is equal to the tangent drawn from A to the circle, and DE is drawn at right angles to AB to meet AC produced in E. Prove that the triangle ADE is equal to the triangle ABC.

55. On the sides of a triangle ABC, points D,E,F are taken, so that AD : DB :: BE : EC :: CF : FA :: 1 : 2. Find the ratio of the triangle DEF to the triangle ABC.

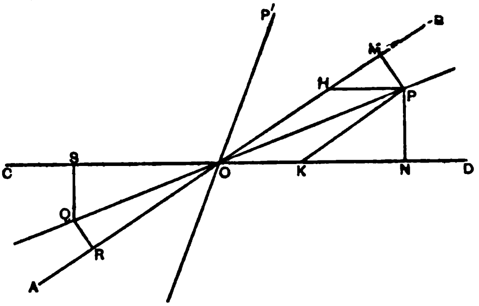
*56. If the vertical angle of a triangle be bisected internally or externally by a straight line which meets the base, the square on the part of the bisector between the vertex and the base is equal to the difference of the rectangle contained by the sides of the triangle and that contained by the segments of the base.

SECTION III.

LOCI AND PROBLEMS.

LOCUS 1. **To find the locus of a point whose distances from two intersecting straight lines are in a given ratio.**

Let AB and CD be two straight lines intersecting at O; it is required to find the locus of a point whose distances from AB and CD are in a given ratio.



Let P be any point such that PM the perpendicular on AB is to PN the perpendicular on CD in the given ratio.

From P draw PH parallel to ON to meet AB at H, and from P draw PK parallel to OM to meet CD at K.

Let P lie within the angle BOD or the vertically opposite angle,
then the angle PHM is equal to the angle PKN, because each is equal to the angle BOD, and the right angles PMH, PNK are equal, therefore the triangles PMH, PNK are similar, *V.* 2.
therefore PH : PM :: PK : PN,
therefore PH : PK :: PM : PN, *Alternando.*
that is PH : PK is equal to a given ratio,
or OK : PK is a given ratio;
and the angle OKP is constant,
because it is supplementary to BOD,
therefore the angle POK is constant, *V.* 3.
therefore P lies on a fixed straight line through O.

In like manner it may be shown that if P′ be a point within the angle BOC or the vertically opposite angle that P′ lies on a second fixed straight line through O.

Also the distances of every point on these two straight lines from AB and CD are in the given ratio.
Let Q be such a point,
draw QR perpendicular to AB,
and QS perpendicular to CD.
Then the triangles QRO, PMO are equiangular,
therefore QR : PM :: OQ : OP, *V.* 2.
and the triangles QSO, PNO are equiangular,
therefore QS : PN :: OQ : OP, *V.* 2.
therefore QR : PM :: QS : PN,
therefore QR : QS :: PM : PN, *Alternando.*
therefore the distances of Q from AB and CD are to one another in the given ratio.

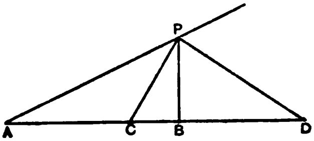
Hence the locus of a point whose distances from AB and CD are in the given ratio is a pair of straight lines through the intersection of AB and CD.

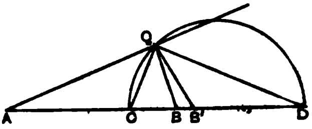
Ex. 57. Prove that OP′ is parallel to HK.

*Ex. 58. The locus of a point whose distances from two parallel straight lines are in a given ratio is a pair of straight lines parallel to the given straight lines.

LOCUS 2. **To find the locus of a point whose distances from two given points are in a given ratio (not one of equality).**

Let A, B be the two given points: it is required to find the locus of a point whose distances from A and B are in a given ratio.





Let P be any point such that PA : PB is equal to the given ratio.

Join AB, and let C and D be the points in which AB is divided internally and externally in the given ratio.
Then, because PA : PB : : AC : CB,
and PA : PB : : AD : DB,
therefore PC and PD are the bisectors of the interior and exterior vertical angles of the triangle APB, *V.* 9.
therefore the angle CPD is a right angle,

therefore P is on the circumference of a circle whose diameter is CD.

Also the distances of every point on the circumference of this circle from A and B are in the given ratio.

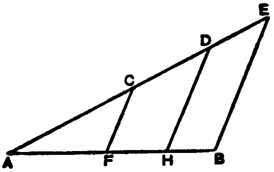
Let Q be any point on the circumference.
Join QA, QB, QC, QD, and make the angle CQB′ equal to the angle CQA.
Then, because QC is the bisector of the angle AQB′, and the angle CQD is a right angle, since it is the angle in a semicircle, therefore QD bisects the exterior vertical angle of the triangle AQB′,
therefore AC : CB′ : : AD : DB′, *V.* 9.
and therefore AC : AD : : CB′ : DB′. *Alternando.*
But AC : CB : : AD : DB,
therefore AC : AD : : CB : DB. *Alternando.*
Therefore CB′ : B′D : : CB : BD,
therefore B′ is the same point as B, *IV.* 10.
therefore QC bisects the angle AQB,
therefore QA : QB : : AC : CB, *V.* 9.
that is, the distances of Q from A and B are to one another in the given ratio.

Hence the locus of a point whose distances from A and B are in the given ratio is the circumference of the circle described on CD, where C and D are the points in which AB is divided internally and externally in the given ratio.

Ex. 59. Find a point C in ACB, a given segment of a circle, such that the straight line AC is double of the straight line CB.

PROB. I. **To divide a straight line similarly to a given divided straight line.**

Let AB be the straight line to be divided,
ACDE the given divided straight line:
it is required to divide AB similarly to ACDE.



Place the straight line AE so as to make any angle with AB.

Join EB, and through C, D draw CF, DH parallel to EB, meeting AB in F and H:
then shall AB be divided at F and H similarly to ACDE.

Because FC, HD and BE are parallel,
therefore AF : FH : : AC : CD,
and FH : HB : : CD : DE, *IV.* 9.
therefore AB is divided at F and H similarly to ACDE.

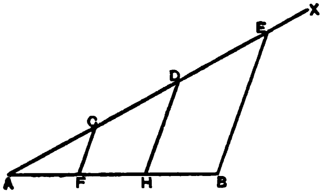
Q.E.F.

PROB. 2. **To divide a straight line internally or externally in a given ratio.**

[For construction and proof see BOOK IV. THEOR. 10.]

PROB. 3. **From a given straight line to cut off any part required.**

Let AB be the given straight line, from which the required part is to be cut off.



Suppose it be required to find the third part of AB.
From A draw a straight line AX making any angle with AB,
in AX take any point C, and make CD and DE each equal to AC, *Postulate* 3.
join EB, and from C draw CF parallel to EB, meeting AB in F :
then shall AF be the third part of AB.

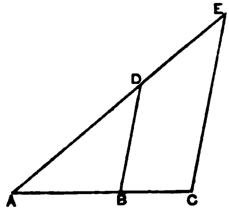
From D draw DH parallel to EB, meeting AB in H :
Because CF, DH and EB are parallel,
and AC, CD, DE are all equal,
therefore AF, FH and HB are all equal, *I.* 34.
therefore AF is the third part of AB.

In like manner we can find the *m*th part of AB whatever whole number *m* may be.

Q.E.F.

PROB. 4. **To find a fourth proportional to three given straight lines.**

Let AB, BC, AD be the three given straight lines:
it is required to find a fourth proportional to AB, BC, AD.



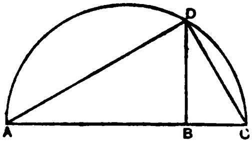
Place AB, BC in the same straight line, making any angle with AD.
Join BD, and from C draw CE parallel to BD and meeting AD produced in E:
then shall DE be the fourth proportional required.

Because BD and CE are parallel,
therefore AB : BC : : AD : DE, *IV.* 9, *Cor.* 2.
that is, DE is a fourth proportional to AB, BC, AD.

Q.E.F.

PROB. 5. **To find a mean proportional between two given straight lines.**

Let AB, BC be the two given straight lines:
it is required to find a mean proportional between AB and BC.



Place AB, BC in the same straight line.
On AC describe the semicircle ADC,

and from B draw BD at right angles to AC, and meeting the circumference at D :
then shall BD be a mean proportional between AB and BC.

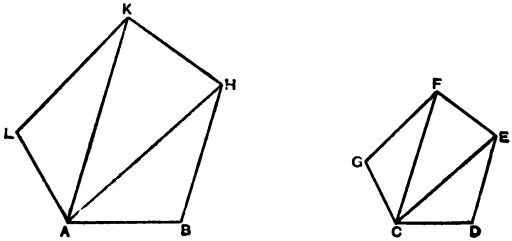
Join AD, CD.

Because ADC is a semicircle,
therefore the angle ADC is a right angle;
and DB is the perpendicular to the hypotenuse of the right-angled triangle DAC,
therefore AB : BD : : BD : BC, *V.* 7, *Cor.*
that is, BD is a mean proportional between AB and BC.

Q.E.F.

PROB. 6. **On a given straight line to describe a rectilineal figure similar and similarly situated to a given rectilineal figure on another given straight line.**

Let AB be the given straight line, CDEFG the given rectilineal figure :
it is required to describe on AB a rectilineal figure similar and similarly situated to CDEFG on CD.



Divide the figure CDEFG into triangles by joining C to E and F.

Make the angles BAH, ABH equal to the angles DCE, CDE respectively,
and make the angles HAK, AHK equal to the angles ECF, CEF respectively,
and so on until triangles have been made equiangular to all the triangles into which the given figure was divided:
then shall ABHKL be the rectilineal figure required.

By construction, the figure ABHKL is equiangular to the figure CDEFG.

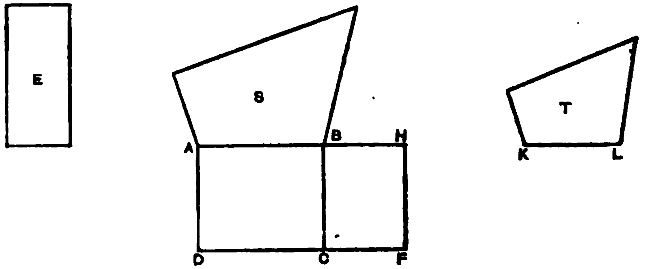
Because the triangles ABH, CDE are equiangular,
therefore AB : BH : : CD : DE, *V.* 2.
and BH : HA : : DE : EC ;
and because the triangles AHK, CEF are equiangular,
therefore HA : HK : : EC : EF ;
therefore BH : HK : : DE : EF, *Ex æquali.*
In like manner HK : KL : : EF : FG,
and so on.
Therefore the sides about the equal angles of the figures are proportional.

Hence the figure ABHKL is similar to the figure CDEFG.
Q.E.F.

Ex. 60. Of all the figures that can be described on the given line similar to the given figure, the greatest has to the least the duplicate ratio of the greatest side of the figure to the least.

PROB. 7. **To describe a rectilineal figure equal to one and similar to another given rectilineal figure.**

Let E and S be two given rectilineal figures:
it is required to describe a rectilineal figure equal to the figure E and similar to the figure S.



On AB a side of S construct the rectangle ABCD equal to the figure S, *II. Prob.* 3.
and on BC construct the rectangle BCFH equal to the figure E.
Take KL a mean proportional between AB and BH, and on KL describe the rectilineal figure T similar to S, so that KL and AB are homologous sides of T and S: *V. Prob.* 6
then shall T be the rectilineal figure required.

Because AB : KL : : KL : BH,
therefore AB is to BH in the duplicate ratio of AB to KL, but the figure S is to the figure T in the duplicate ratio of AB to KL,
therefore the figure S is to the figure T as AB is to BH.
And because rectangles of equal altitude are to one another as their bases,
therefore the rectangle AC is to the rectangle BF as AB is to BH.
Therefore the figure S is to the figure T as the rectangle AC is to the rectangle BF.
But the figure S is equal to the rectangle AC,

therefore the figure T is equal to the rectangle BF,
and therefore to the given figure E.
Therefore T is the rectilineal figure required.

Q.E.F.

EXERCISES.

61. Through a given point A draw a line meeting the given lines OX, OY in points P, Q respectively, so that OP is to OQ in a given ratio.

62. Through two fixed points in the circumference of a circle draw two parallel chords which shall be to each other in a given ratio.

63. Construct a triangle similar to a given triangle, having one vertex fixed and the others on two given straight lines.

64. Draw from a given point P two straight lines PQ, PR at a given inclination to one another to meet two given straight lines in Q and R so that PQ,PR may be equal.

65. Describe an isosceles triangle equal to a given triangle and having a given vertical angle.

66. Three straight lines meet in a point. Draw a fourth cutting them so that the intercepted segments may have given magnitudes.

67. In a given square inscribe a square of which the area shall be equal to three-fourths of that of the given square.

EXERCISES ON BOOK V.

68. The perpendiculars from the vertices of a triangle upon the opposite sides are inversely proportional to the sides to which they are respectively perpendicular.

69. If in the same circle there be inscribed two triangles of equal area, then the rectangle contained by any two sides of the one is to the rectangle contained by any two sides of the other, as the base of the second is to the base of the first.

70. Produce a given straight line so that the rectangle contained by it and the produced part shall be equal to a given square.

71. From the same point A straight lines are drawn, making the angles BAC, CAD, DAE equal to one another, and they are cut by a straight line BCDE, which makes BAE an isosceles triangle: shew that BC or DE is a mean proportional to BE and CD.

72. Given the base of a triangle, and the point where the line bisecting the exterior vertical angle cuts the base produced, find the locus of the vertex of the triangle.

73. P is any point in the circumference of a circle whose centre is O. Join OP, and produce it to Q, making PQ equal to n times OP, and from Q draw a straight line touching the circle in R. Join PR, and shew that the diameter of the circle described about the triangle PQR is equal to $(n + 1)$ PR.

74. ABD is the diameter of a semicircle ACD, and ABC is a right angle. E, any point on the chord AC within the semicircle, is joined to B, and CF is drawn cutting AD in F,

and making the angle BCF equal to the angle ABE. Prove that AE is to EC as BF is to BD.

75. Find the locus of a point such that the tangents from it to each of two given circles may contain equal angles.

76. If a quadrilateral ABCD, having the sides BC, CD equal be inscribed in a circle, the rectangle AB, AD, together with the square on BC, will be equal to the square on AC.

77. On a level plain are to be seen two church spires; a man walks on the plain so that he always sees the spires at equal angles of elevation. Prove that he walks in a circle.

78. ABC and ABF are triangles on the same base in the ratio of 2 to 1; AF and BF produced meet BC and AC in D and E; in FB, FG is cut off equal to FE, and BG is bisected in O. Prove that BO : BE :: DF : DA.

79. From each of two opposite angular points of a parallelogram lines are drawn to the points of bisection of the sides containing the opposite angle. Shew that these lines will form a parallelogram whose area will be one-third of that of the whole parallelogram.

80. On a given straight line construct a right-angled triangle whose three sides shall be in continued proportion.

81. ABC is an isosceles triangle whose vertex is A. From BA, CA cut off BD, CE, each equal to BC, and join BE, CD, cutting each other in F. Shew that if AC is equal to 25 BC, it is also equal to 35 FC.

82. Describe a triangle on a given base, and with a given vertical angle, such that the base may be a mean proportional between the sides. Shew that the problem is impossible, if the given angle is greater than the angle of an equilateral triangle.

DEFINITIONS OF BOOK V.

Def. 1. *If the angles of a rectilineal figure, taken in order, are equal respectively to those of another, also taken in order, the figures are said to be* **equiangular.** *Each angle of the one is said to* **correspond** *to the angle equal to it in the other, and the sides joining the vertices of corresponding angles are termed* **corresponding** *sides.*

Def. 2. **Similar** *figures are such as are equiangular, and have their sides proportional, the corresponding sides being homologous.*

Def. 3. *Similar figures are said to be* **similarly situated upon given straight lines,** *when those straight lines are corresponding sides of the figures.*

Def. 4. *The point of intersection of all straight lines which join the corresponding points of two similar figures, whose corresponding sides are parallel, is called the* **centre of similarity** *of the two figures.*

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